# MORTGAGES

According to the Canada Interest Act, the effective rate charged can not be greater than that which results from semi-annual compounding of the annual percentage rate. i.e. An interest rate quoted as $r_a$ = 8 % implies an effective yield of 8.16 % per year. Since payments are generally made monthly, a period interest rate needs to be calculated.

$(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$

$r = (1+0.08/2)^{1/6} - 1 \cong 0.65582\ \%$

Thus if $r_a = 0.08$, $r = 0.0065582$ is the effective monthly rate.

Like finite coupon bonds (no principal payment), mortgages are a stream of payments made monthly, and the calculations are identical.

$P_0 = C/r - (C/r)\,[(1 + r)^{-T}]$ or solving for $\mathbf{C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}}$

where $P_0$ is the present value of the mortgage or amount of mortgage

C is the mortgage payment

r is the period interest rate calculated above, and

T is the number of periods over which the mortgage is amortized (generally 25 years with 12 months per year = 300)

e.g. What monthly payment must be made on a \$100 000 mortgage at $r_a$ = 8 %, with a 25 year term (term is the length of the contract), and a 25 year amortization? Amortization means "gradual extinguishing by money put aside." Here it is the period to repay the loan.

C = \$763.22 Actually, the calculation yields $\dfrac{(\$100000)(0.0065582)}{1-(1.0065582)^{-300}} = \$763.2137$

but the convention is that mortgage payments are always rounded up and a smaller final payment is made to compensate.

## EXAMPLES

The Johnsons have a $50 000 mortgage with monthly payments over 25 years at interest rate $r_a$ = 10 %. Since their employer pays them weekly, they decide to switch to weekly payments (at the same interest rate and term). Find and compare their weekly payments to monthly payments on the same mortgage.

Solution:

Monthly Payment

$(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$

$r = (1+0.10/2)^{1/6} - 1 \cong 0.81648$ %

$C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}$ $\quad \therefore C = \dfrac{(50000)(0.0081648)}{1 - (1.0081648)^{-300}}$ $\quad \therefore$ **C = $ 447.25**

Weekly Payment

$(1 + r)^{52} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/26} - 1$

$r = (1+0.10/2)^{1/26} - 1 \cong 0.18783$ %

$C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}$ $\quad \therefore C = \dfrac{(50000)(0.0018783)}{1 - (1.0018783)^{-1300}}$ $\quad \therefore$ **C = $ 102.89**

**Note a monthly savings of $ 35.69 by paying weekly over the monthly option**

Mr. Freemason requires a $60 000 mortgage for his new house and the bank offers him one at $r_a$ = 10.25 %.

a) Find his monthly mortgage payment based on the following repayment periods: 25 years, 20 years and 15 years.

b) If Mr. Freemason can afford to pay at most $950 monthly on his mortgage, what repayment period (in whole numbers of years) should he request?

Solution:

a) 25 years: $(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$

$r = (1+0.1025/2)^{1/6} - 1 \cong 0.83648$ %

$C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}$ $\quad \therefore C = \dfrac{(60000)(0.0083648)}{1 - (1.0083648)^{-300}}$ $\quad \therefore$ **C = $ 546.82**

20 years: $(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$

$r = (1+0.1025/2)^{1/6} - 1 \cong 0.83648$ %

$C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}$ $\quad \therefore C = \dfrac{(60000)(0.0083648)}{1 - (1.0083648)^{-240}}$ $\quad \therefore$ **C = $ 580.52**

15 years: $(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$
$r = (1+0.1025/2)^{1/6} - 1 \cong 0.83648\ \%$

$C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}$ $\therefore\ C = \dfrac{(60000)(0.0083648)}{1 - (1.0083648)^{-180}}$ $\therefore$ **C = $ 646.15**

b) $(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$
$r = (1+0.1025/2)^{1/6} - 1 \cong 0.83648\ \%$

$C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}$ $\therefore\ 950 = \dfrac{(60000)(0.0083648)}{1 - (1.0083648)^{-T}}$

$\therefore\ 950 - 950(1.0083648)^{-T} = 501.8880$ $\therefore\ \ln(950) - T \ln(1.0083648) = \ln(448.112)$
$\therefore\ T = 90.2062$ months $\cong 7.52$ years $\therefore$ **He should request an 8 year repayment**

Mr. and Mrs. Jones are considering the purchase of a new house. They will require a mortgage of $35 000 at rate $r_a = 9.75\ \%$. If they can afford to pay $900 monthly, how many full payments will be required, and what will the final smaller payment be?

Solution:
$(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$
$r = (1+0.0975/2)^{1/6} - 1 \cong 0.79647\ \%$

$C = \dfrac{(P_0 r)}{1 - (1 + r)^{-T}}$ $\therefore\ 900 = \dfrac{(35000)(0.0079647)}{1 - (1.0079647)^{-T}}$

$\therefore\ 900 - 900(1.0079647)^{-T} = 278.7645$ $\therefore \ln(900) - T \ln(1.0079647) = \ln(621.2355)$
$\therefore\ T = 46.726$ months ` $\therefore$ **They should make 46 full payments**

$C_f = [P_0 (1 + r)^T] - C\ \left[\dfrac{(1 + r)^{T-1} - 1}{r}\right] [(1+r)]$

$C_f = [35000\ (1.0079647)^{47}] - 900\ \left[\dfrac{(1.0079647)^{46} - 1}{0.0079647}\right] [1.0079647]$

**$C_f$ = $ 654.13**

A family buys a house worth $326 000. They pay $110 000 down and then take out a 5 year mortgage for the balance at $r_a = 10.5\ \%$ to be amortized over 20 years. Payments will be made monthly. Find the outstanding principal balance at the end of 5 years and the owners' equity at that time.

Solution:
$(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$
$r = (1+0.105/2)^{1/6} - 1 \cong 0.85645\ \%$

$$C = \frac{(P_0 r)}{1 - (1+r)^{-T}} \quad \therefore C = \frac{(216000)(0.0085645)}{1 - (1.0085645)^{-240}} \quad \therefore \mathbf{C = \$\ 2124.30}$$

Consider: $B_k = [P_0 (1 + r)^k] - C \left[ \frac{(1 + r)^k - 1}{r} \right]$ ,

where $B_k$ is the balance at the end of the $k^{th}$ period (after the $k^{th}$ payment).

$$B_{60} = [216000 (1.0085645)^{60}] - 2124.30 \left[ \frac{(1.0085645)^{60} - 1}{0.0085645} \right] \quad \therefore \mathbf{B_{60} = \$\ 194597.26}$$

**Owners' Equity = 110000 + 216000 - 194597.26 = $ 131402.74**

A couple buys a house and assumes a $90 000 mortgage to be amortized over 25 years. The interest rate is guaranteed at $r_a = 8$ %. The mortgage allows the couple to make extra payments against the outstanding principal each month. By saving carefully, the couple manages to pay off an extra $100 each month. Because of these extra payments, how long will it take to pay off the mortgage and what will be the amount of the final smaller monthly payment?

Solution:

$(1 + r)^{12} = [1 + (r_a/2)]^2$ or $r = [1 + (r_a/2)]^{1/6} - 1$

$r = (1+0.08/2)^{1/6} - 1 \cong 0.65582$ %

$$C = \frac{(P_0 r)}{1 - (1+r)^{-T}} \quad \therefore C = \frac{(90000)(0.0065582)}{1 - (1.0065582)^{-300}} \quad \therefore \mathbf{C = \$\ 686.90}$$

$$B_k = [P_0 (1 + r)^k] - C \left[ \frac{(1 + r)^k - 1}{r} \right]$$

$$0 = [90000 (1.0065582)^k] - 786.90 \left[ \frac{(1.0065582)^k - 1}{0.0065582} \right]$$

$\therefore$ $0 = 90000 (1.0065582)^k - 119987.1916(1.0065582)^k + 119987.1916$

$\therefore$ $29987.1916 (1.0065582)^k = 119987.1916$

$\therefore$ $\ln (29987.1916) + k \ln (1.0065582) = \ln (119987.1916)$

**$\therefore$ k = 212.1248 months $\therefore$ it will take 17 years and 9 months with smaller final payment to pay off the mortgage**

$$C_f = [P_0 (1 + r)^T] - C \left[ \frac{(1 + r)^{T-1} - 1}{r} \right] [(1+r)]$$

$$C_f = [90000 (1.0065582)^{213}] - 786.90 \left[ \frac{(1.0065582)^{212} - 1}{0.0065582} \right] [1.0065582]$$

**$C_f$ = $98.45**