Introduction to Quantum Computing
Assignment 1 - Due Jan. 22
Stern-Gerlach experiment and Dirac's notation

1. **Alternative basis for the 1 qubit problem.**– Show that $\{|S_y,+\rangle, |S_y,-\rangle\}$ provides a new complete basis for the 1 qubit Hilbert space. To keep the notation concise you can denote $|S_z,+\rangle \equiv |0\rangle$ and $|S_z,-\rangle \equiv |1\rangle$.
   Hint: You have to show that (a) this new basis is orthonormal and (b) it satisfies the completeness relation.

2. **Change of basis.**– Write $|S_x,+\rangle$ in the $\{|S_y,+\rangle, |S_y,-\rangle\}$ basis.

3. **Pauli matrices.**– The spin operator $\hat{S}_x$ is equal to $\frac{\hbar}{2}\hat{X}$, where $\hbar$ is called "Planck's constant" ($\hbar$ has units of angular momentum) and $\hat{X}$ is an operator called "Pauli-X" matrix. It has the following property: Its eigenvectors are $|S_x,+\rangle$ (eigenvalue $+1$) and $|S_x,-\rangle$ (eigenvalue -1). In Dirac's notation this means

$$\hat{X} = |S_x,+\rangle\langle S_x,+| - |S_x,-\rangle\langle S_x,-|. \tag{1}$$

   (a) Write $\hat{X}$ in explicit matrix form.
   (b) Following the same logic, write $\hat{Y}$ in explicit matrix form.
   (c) Write $\hat{Z}$ in explicit matrix form (it's the easiest one!).

4. **Stern-Gerlach states for arbitrary spin direction.**– Following the analogy with polarization of light described in class, obtain an expression that relates the states $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},+\rangle$ and $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},-\rangle$ to the states $|S_z,+\rangle \equiv |0\rangle$ (spin up) and $|S_z,-\rangle \equiv |1\rangle$ (spin down). Here $\hat{\boldsymbol{n}}$ is an arbitrary unit vector in spherical coordinates:

$$\hat{\boldsymbol{n}} = \cos(\phi)\sin(\theta)\hat{\boldsymbol{x}} + \sin(\phi)\sin(\theta)\hat{\boldsymbol{y}} + \cos(\theta)\hat{\boldsymbol{z}}. \tag{2}$$

   Hint: Argue that $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},+\rangle$ should be equivalent to light polarized along the direction

$$\hat{\boldsymbol{n}}' = \cos\left(\frac{\theta}{2}\right)\hat{\boldsymbol{x}} + \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\phi}\hat{\boldsymbol{y}}. \tag{3}$$

   Also argue that the $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},-\rangle$ can be obtained from $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},+\rangle$ by $\theta \to \pi - \theta$ and $\phi \to \phi + \pi$ (that's equivalent to "flipping" the spin from $+\hat{\boldsymbol{n}}$ into $-\hat{\boldsymbol{n}}$).

5. **Pauli matrix in arbitrary direction.**– You will now be able to check whether the $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},\pm\rangle$ states you obtained in problem 4 above are the correct ones.

   (a) Using your results in problem 3, write down the Pauli matrix in arbitrary direction $\hat{\boldsymbol{n}}$ in explicit matrix form:

   $$\hat{\sigma}_n = \cos(\phi)\sin(\theta)\hat{X} + \sin(\phi)\sin(\theta)\hat{Y} + \cos(\theta)\hat{Z}. \tag{4}$$

   (b) Show with explicit calculation that your $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},+\rangle$ obtained in problem 4 is an eigenvector of $\hat{\sigma}_n$ with eigenvalue $+1$.

   (c) Show with explicit calculation that your $|\boldsymbol{S}\cdot\hat{\boldsymbol{n}},-\rangle$ obtained in problem 4 is an eigenvector of $\hat{\sigma}_n$ with eigenvalue -1.