Introduction to Quantum Computing
Assignment 5 - Due March 26
Quantum algorithms with IBM-Q

Instructions:

• All problems below should be solved with Qiskit using a Jupyter notebook. Once you’re done, submit your .ipynb file(s) to the A5 area in BrightSpace.

1. Quantum cryptography.– Write a Qiskit code that implements BB84 with 15 qubits. You will play Alice.

   (a) First use the local simulator qasm_simulator to play Bob. State clearly the value of the one-time pad that you got.

   (b) Second, use the real device ibmq_16_melbourne to play Bob. Again, clearly state the one-time pad that you got.

2. Quantum cloning.

   (a) Design a quantum circuit that clones the states \(|+\rangle\) and \(|-\rangle\). I.e., design a unitary \(U\) that achieves the following:

   \[
   U \left( |+\rangle |0\rangle \right) = |+\rangle |+\rangle, \\
   U \left( |-\rangle |0\rangle \right) = |-\rangle |-\rangle. 
   \]  

   (b) Apply this \(U\) to \(|0\rangle\) and \(|1\rangle\). Evaluate the “cloning fidelity” by computing

   \[
   F_\psi = |\langle \psi | U (|\psi\rangle |0\rangle) |^2, 
   \]

   with the qasm_simulator, for \(|\psi\rangle = |0\rangle\) and \(|\psi\rangle = |1\rangle\). How good is the cloning?

3. One-qubit tomography.– Write the three circuits necessary to measure the projections of the state

   \[
   |\psi\rangle = \cos \left( \frac{\pi}{8} \right) |0\rangle + \sin \left( \frac{\pi}{8} \right) |1\rangle, 
   \]

   along the \(x\), \(y\), and \(z\) axes of the Bloch sphere. Do this for the simulator and for a real device using 1000 shots. Use the function plot_bloch_vector([px,py,pz]) to display the Bloch vectors. How do they compare?
4. **Error correction with the three-qubit bit-flip code.**— The circuit shown below is the “automated” version of the three-qubit bit-flip error correction code described in class. The last five gates – Two CNOTS and three Toffolis – perform the same “correction protocol” achieved by the conditional measurement operations shown in class. The final state $|\Phi\rangle$ of the ancillas remains $|00\rangle$ if no error happens, and becomes $|10\rangle$, $|11\rangle$, $|01\rangle$ when one of the qubits 2, 1, 0 flips, respectively. If another round of error correction is needed the ancilla state $|\Phi\rangle$ must be reset to $|00\rangle$ (currently not allowed in real IBM-Q devices).

(a) Implement this algorithm for

$$|\psi_{in}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$$

in the simulator and check that it works by intentionally flipping each one of the three qubits in $|\psi_{in}\rangle$ before the error correction procedure. Measure $|\psi_{out}\rangle$ in the computational basis and check that you get close to 50% probability for 000 and 111.

(b) Run the same circuit in a real device (no need to add the intentional flips!), and measure $|\psi_{out}\rangle$. How good is it?

(c) As a comparison, run the same algorithm in the same real device *without* the last five “correction protocol” gates. How does the computational basis probabilities for $|\psi_{out}\rangle$ compare to the ones found in item (b)? Conclude by saying whether you think the device you used has low enough noise to take advantage of error correction.
5. **Three-bit phase estimation.**

(a) Implement the circuit below for $U = R_y(\theta)$ with $\frac{\theta}{4\pi}$ exactly represented by three base-2 decimals, $\frac{\theta}{4\pi} = 0.j_1j_2j_3$ (your choice of $j_1j_2j_3$). Note that $\frac{\theta}{4\pi}$ is the associated phase $\phi_u$ for $|u\rangle = |-i\rangle$, since

$$R_y(\theta) |-i\rangle = e^{+i\frac{\theta}{2}} |-i\rangle = e^{+2\pi i \frac{\theta}{4\pi}} |-i\rangle.$$  \hspace{1cm} (5)

Input $|\psi\rangle = |-i\rangle$, and run the circuit in the simulator and a real device to compare.

(b) Run the circuit again for $\frac{\theta}{4\pi} = 0.j_1j_2j_3 - \epsilon$, with $\epsilon$ small. If you run the circuit many times does it return the $0.j_1j_2j_3$ closest to the correct answer $\frac{\theta}{4\pi}$? How does the real device compare to the simulator?

\[
\begin{array}{c}
|0\rangle \rightarrow H \rightarrow H \rightarrow R_x(-\frac{\pi}{4}) \rightarrow R_x(-\frac{\pi}{2}) \rightarrow j_1 \\
|0\rangle \rightarrow H \rightarrow H \rightarrow R_x(-\frac{\pi}{2}) \rightarrow j_2 \\
|0\rangle \rightarrow H \rightarrow H \rightarrow j_3 \\
|\psi\rangle \rightarrow U \rightarrow U^2 \rightarrow U^4 \rightarrow |u\rangle
\end{array}
\]