Qubits and Linear Algebra with Dirac's notation
Survey of complex numbers, linear algebra, and Dirac notation

What we learned from S-Q: Quantum two-level systems (e.g., spin of the electron) exist in nature

Type of electron is a "naturally" split
\[
\begin{bmatrix} S_x \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} S_y \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} S_z \end{bmatrix} = \frac{\hbar}{2} \begin{bmatrix} 1 \end{bmatrix}
\]

- If \( S_z = \frac{\hbar}{2} \begin{bmatrix} 1 \end{bmatrix} \) state of the system, we call it spin-up (\( \uparrow \))
- If \( S_z = \frac{\hbar}{2} \begin{bmatrix} -1 \end{bmatrix} \) state of the system, we call it spin-down (\( \downarrow \))

- Both states are orthogonal (\( \downarrow \) and \( \uparrow \))
- Probes in spin states: \( \uparrow \) and \( \downarrow \) are eigenstates of \( S_z \)
- \( \uparrow \) is an eigenstate of \( S_z \) with eigenvalue \( +\frac{\hbar}{2} \)
- \( \downarrow \) is an eigenstate of \( S_z \) with eigenvalue \( -\frac{\hbar}{2} \)

But most quantum systems are NOT two-level systems!

- Not just current QC’s; use for any qubit-based devices requiring high-fidelity operations (MSQD and quantum computing)
- Use in quantum cryptography and quantum entanglement
- Quantum mechanical concepts of polarization and interference
- The qubit state can be in a superposition of states

Crash course on complex numbers

\[ a = \alpha + \beta i \quad (x, y \in \mathbb{R}) \]

- Complex conjugation and modulus:
\[ |a|^2 = \alpha^2 + \beta^2 \]

- Representation using polar form and Euler’s identity
\[ a = \alpha + \beta i = \rho \cos \theta + \rho \sin \theta i = \rho e^{i \theta} \]

Exponentials and Euler’s identity

\[ e^{i \theta} = \cos \theta + i \sin \theta \]

Linear algebra with Dirac’s notation

- A simple vector space \( \mathbb{R}^2 \)

\[ \begin{bmatrix} \alpha \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \]

Inner and outer products

- Inner or "dot" product
\[ a \cdot b = \sum_{i=1}^{n} a_i b_i \]

- Outer or "tensor" product
\[ a \otimes b = \sum_{i=1}^{n} a_i b_i \]

Basis and completeness

- The qubit state space of linear product is called "Hilbert space". All states in this space can be written as a linear combination of a set of special basis called "basis set".

Let’s see if we got this

\[ \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} = \begin{bmatrix} 2 \end{bmatrix} \]

Summary

- Dirac’s notation provides a convenient way for dealing with states (complex-valued states) in quantum theory.
- Hilbert space formalism is very useful for understanding quantum mechanics.