Qubits and Linear Algebra with Dirac's notation

Survey of complex numbers, linear algebra, and Dirac notation

What we learned from S-G: Quantum two-level systems (e.g. spin of the electron) exist in nature

Spin of electron is a "natural" *qubit*

$$|S_z,+\rangle \equiv |0\rangle$$
$$|S_z,-\rangle \equiv |1\rangle$$

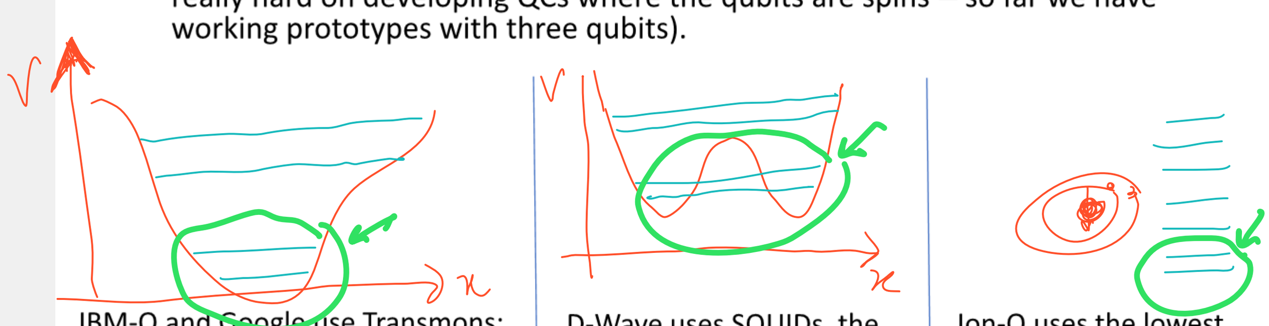
- Qubit = Quantum two-level system. S-G shows that they can be in any state described by

$$\boxed{|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \qquad \alpha,\beta \in \mathbb{C}}$$

- We say that "spin is quantized". If we "read-out" the state of the spin qubit we will get "0" with probability $|\alpha|^2$ and "1" with probability $|\beta|^2$

But most quantum systems are NOT two-level systems!

- In fact, current QCs do not use the spin as a qubit (people are working really hard on developing QCs where the qubits are spins – so far we have working prototypes with three qubits).



IBM-Q and Google use Transmons: This qubit is formed by the lowest two energy levels of a potential well

D-Wave uses SQUIDs, the lowest two energy levels of a double well.

Ion-Q uses the lowest two energy levels of a atom with many spins.

Crash course on complex numbers

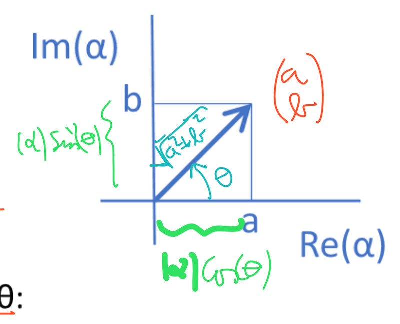
$$i^2 = -1$$

$$\boxed{\alpha = a + ib \ \ (a, b \in \mathbb{R})}$$

- Complex conjugation and modulus:

$$\alpha^* = a - ib$$

$$|\alpha|^2 = \alpha^*\alpha = (a-ib)(a+ib)$$
$$= a^2 + \cancel{iab} - \cancel{iba} + b^2$$
$$= a^2 + b^2$$



- Represent using modulus and phase θ:

$$\alpha = a + ib = |\alpha|(\cos\theta + i\sin\theta) = |\alpha| e^{i\theta}$$

PHASE

Exponentials and Euler's identity

Real number

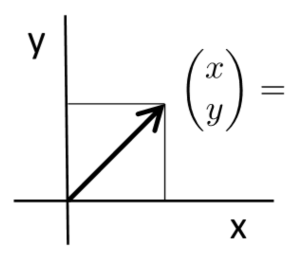
$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{n'=0}^{\infty} \frac{(i\theta)^{2n'}}{(2n')!} + \sum_{n'=0}^{\infty} \frac{(i\theta)^{2n'+1}}{(2n'+1)!}$$

$$(i\theta)^{2n'} = (i^2)^{n'}\theta^{2n'} = (-1)^{n'}\theta^{2n'} \qquad (i\theta)^{2n'+1} = i(-1)^{n'}\theta^{2n'+1}$$

$$= \underbrace{\sum_{n'=0}^{\infty} \frac{(-1)^{n'}\theta^{2n'}}{(2n')!}}_{=\cos(\theta)} + i \underbrace{\sum_{n'=0} \frac{(-1)^{n'}\theta^{2n'+1}}{(2n'+1)!}}_{\sin(\theta)} = \cos(\theta) + i\sin(\theta)$$

Linear algebra with Dirac's notation

- A simple vector space: $\mathbb{R}^2$



$$\begin{pmatrix} x \\ y \end{pmatrix} = x\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=|0\rangle} + y\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=|1\rangle}$$

- The vector space for one qubit, $\mathbb{C}^2$. This is where the "kets" live:

$$|\Psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \underbrace{|0\rangle}_{\equiv\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + \beta \underbrace{|1\rangle}_{\equiv\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

- The qubit's "co-vector" or dual space, the "bras" live:

$\dagger \to$ "DAGGER" = * + transpose

$$\langle\Psi| = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}^{\dagger} = (\alpha^* \ \beta^*) = \alpha^* \underbrace{(1 \ 0)}_{\langle 0|} + \beta^* \underbrace{(0 \ 1)}_{\langle 1|} = \alpha^*\langle 0| + \beta^*\langle 1|$$

Inner and outer products

- Inner or "dot" product:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \qquad |\varphi\rangle = \begin{pmatrix} \gamma \\ \delta \end{pmatrix}$$

DOT PRODUCT

$$\underbrace{(\alpha^* \ \beta^*)}_{\langle\psi|} \cdot \underbrace{\begin{pmatrix} \gamma \\ \delta \end{pmatrix}}_{|\varphi\rangle} = \alpha^*\gamma + \beta^*\delta = \langle\psi|\varphi\rangle$$

MODULUS SQUARED OF $|\psi\rangle$:

1 QUBIT

$$\langle\psi|\psi\rangle = (\alpha^* \ \beta^*)\cdot\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |\alpha|^2 + |\beta|^2 \stackrel{?}{=} 1$$

- Outer or "tensor" product:

FOR SHORT

$$|\psi\rangle \otimes |\varphi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \\ \beta\begin{pmatrix} \gamma \\ \delta \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix} = |\psi\rangle|\varphi\rangle = |\psi\varphi\rangle$$

Basis and completeness

- The qubit vector space with inner product is called "Hilbert space". All states in this space can be written as a linear combination of a set of special kets called "basis kets". $\{|0\rangle, |1\rangle\}$

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$\left.\begin{matrix} \langle 0|1\rangle = 0 \\ \langle 0|0\rangle = \langle 1|1\rangle = 1 \end{matrix}\right\} \text{ORTHONORMALITY!}$$

$$\{|j\rangle\}_{j=0}^{2^n-1}$$

$$|\psi\rangle = \sum_j \alpha_j |j\rangle \quad \text{FOR ANY } |\psi\rangle$$

- A set of *orthonormal* kets is a *basis* when it satisfies the *completeness* (or closure) relation.

$$\sum_{j=0}^{2^n-1} |j\rangle\langle j| = \mathbb{1}$$

FOR 1 QUBIT

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$$

CHECK: TAKE ARBITRARY $|\psi\rangle$:

$$(|0\rangle\langle 0| + |1\rangle\langle 1|)|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$= \alpha(|0\rangle\underbrace{\langle 0|0\rangle}_{=1} + |1\rangle\cancel{\langle 1|0\rangle}) + \beta(|0\rangle\cancel{\langle 0|1\rangle} + |1\rangle\underbrace{\langle 1|1\rangle}_{=1})$$

$$= \alpha|0\rangle + \beta|1\rangle$$
$$= \mathbb{1}|\psi\rangle \ !!!$$

$$|0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes (1 \ 0) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes (0 \ 1) =$$

$$\begin{pmatrix} 1\,(1 \ 0) \\ 0\,(1 \ 0) \end{pmatrix} + \begin{pmatrix} 0\,(0 \ 1) \\ 1\,(0 \ 1) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(1 \ 0)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\cdot 0 + 0 \cdot 1 = 0$$

Let's see if we got this

- Split up into groups to write the following bra-ket expressions in number, vector, or matrix form:

a) $\langle 0|1\rangle = 0$

b) $|0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

c) $\underbrace{|0\rangle|1\rangle}_{|01\rangle} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

d) $\hat{Y} = |S_y,+\rangle\langle S_y,+| - |S_y,-\rangle\langle S_y,-| = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\underbrace{|10\rangle}_{=2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

From last class: $|S_y, \pm\rangle = \dfrac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$

Summary

- Dirac's notation provides a convenient way for dealing with states (vectors) and operators (matrices) in quantum theory.

- Next class we will talk about operators, observables, and probabilities of outcomes of measurements.