Quantum measurement

Trace operation

- The trace of a matrix is the sum of its diagonals

$$\mathrm{Tr}\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} = 1+4=5$$

$\hat{A} = |0\rangle\langle 0|\hat{A}|0\rangle\langle 0| + |0\rangle\langle 0|\hat{A}|1\rangle\langle 1| + \dots$
$= \sum_{i,j=0,1} A_{ij}|i\rangle\langle j|$
$= |+\rangle\langle +|\hat{A}|+\rangle\langle +| + |+\rangle\langle +|\hat{A}|-\rangle\langle -| + \dots$

- Trace is extremely useful because it is always the same irrespective of the basis you used to represent your matrix! Proof:

$\mathrm{Tr}\{\hat{A}\} = \sum_\alpha \langle\alpha|\hat{A}|\alpha\rangle = \sum_{\alpha,\beta}\langle\alpha|\hat{A}|\beta\rangle\langle\beta|\alpha\rangle = \sum_{\beta,\alpha}\langle\beta|\alpha\rangle\langle\alpha|\hat{A}|\beta\rangle$

$\{|\alpha\rangle\}$ IS A BASIS

$\hat{I} = \sum_\beta |\beta\rangle\langle\beta|$

$\{|\beta\rangle\}$ IS ANOTHER BASIS!

$= \sum_\beta \langle\beta|\underbrace{\left(\sum_\alpha |\alpha\rangle\langle\alpha|\right)}_{=\hat{I}}\hat{A}|\beta\rangle$

$= \sum_\beta \langle\beta|\hat{A}|\beta\rangle$

$\hat{X} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ WHEN REPRESENTED IN BASIS $\{|0\rangle, |1\rangle\}$

$\hat{X} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ IN BASIS $\{|+\rangle, |-\rangle\}$.

$\{|\pm\rangle\}$

$\mathrm{Tr}\{\hat{X}\} = 0+0 = 0$ IN $\{|0\rangle, |1\rangle\}$
$= 1-1 = 0$ IN $\{|+\rangle, |-\rangle\}$

Quantum measurement in the computational basis

- Each time a qubit is measured, the outcome is either 0 or 1, and the state of the qubit *collapses* to $|0\rangle$ or $|1\rangle$:

$$|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \;-\!\boxed{\mathbf{M}}\!\rightarrow |x\rangle \begin{cases} x=0 \text{ with probability } |\langle 0|\Psi\rangle|^2 = |\alpha_0|^2 \\ x=1 \text{ with probability } |\langle 1|\Psi\rangle|^2 = |\alpha_1|^2 \end{cases}$$

$|\alpha_0|^2 + |\alpha_1|^2 = 1$

- For n qubits:

$$|\Psi\rangle = \sum_{j=0}^{2^n-1}\alpha_j|j\rangle \;-\!\boxed{\mathbf{M}}\!\rightarrow |j\rangle \text{ with prob. } |\langle j|\Psi\rangle|^2 = |\alpha_j|^2$$

- This implies normalization: $\sum_j \underbrace{|\langle j|\psi\rangle|^2}_{P(j)} = 1 = \sum_j \langle\psi|j\rangle\langle j|\psi\rangle = \langle\psi|\psi\rangle = 1$

NORM OF $|\psi\rangle = 1$: $\boxed{\langle\psi|\psi\rangle = 1}$

Projectors

- Projector: Hermitian operator satisfying $\boxed{\hat{P}^2 = \hat{P}.}$ Examples:

$\hat{P}_0 = |0\rangle\langle 0|$ $\quad \hat{P}_0^2 = (|0\rangle\langle 0|)(|0\rangle\langle 0|) = |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0| = \hat{P}_0$

$\hat{P}_\Psi = |\psi\rangle\langle\psi| \equiv \hat{\rho}$ "DENSITY MATRIX" OR "DENSITY OPERATOR".

FOR QUBIT IN STATE $|\psi\rangle$.

- Consider a basis $\{|a\rangle\}$. The set of projectors $\hat{P}_a = |a\rangle\langle a|$ satisfies:

COMPLETENESS:
$$\sum_a \hat{P}_a = \hat{I}$$

$\sum_a |a\rangle\langle a| = \hat{I}$

Projective measurements

- A projective measurement of state $|\Psi\rangle$ in basis $\{|a\rangle\}$ detects whether the qubit is in one of the members of the basis:

$$p(a) = |\langle a|\Psi\rangle|^2$$

$$= \mathrm{Tr}\left\{\hat{P}_a\,\hat{\rho}\right\} = \mathrm{Tr}\{|a\rangle\langle a|\,|\psi\rangle\langle\psi|\}$$

$= \sum_b \langle b|(|a\rangle\langle a|\,|\psi\rangle\langle\psi|)|b\rangle = \sum_b \underbrace{\langle b|a\rangle}\langle a|\psi\rangle\langle\psi|b\rangle$

$= \delta_{b,a} = \begin{cases}1 & \text{WHEN } a=b\\ 0 & \text{WHEN } a\neq b\end{cases}$

$= \langle a|\psi\rangle\langle\psi|a\rangle = |\langle a|\psi\rangle|^2$

TRACE IN BASIS $\{|b\rangle\} = \{|a\rangle\}$

$\sum_b \delta_{b,a} f(b) = f(a)$

DISCRETE ANALOG OF

$\int dx\, \delta(x-a) f(x) = f(a)$

- Say the outcome of the measurement is $|a\rangle \in \{|a\rangle\}$. In this case the qubit collapses to the state $|a\rangle$!

$$|\Psi\rangle \;-\!\boxed{\underset{\{|a\rangle\}}{\mathbf{M}}}\!\rightarrow |a\rangle \text{ with prob. } |\langle a|\Psi\rangle|^2$$

"COLLAPSES" TO $|a\rangle$!

Let's practice with 2 qubits

Consider the two qubit state:

$$|\Psi\rangle = \alpha_0|00\rangle + \alpha_1|01\rangle + \alpha_2|10\rangle + \alpha_3|11\rangle$$

$|01\rangle$ or $|10\rangle$

1) What is the probability of measuring the 1st qubit in a DIFFERENT state of the 2nd qubit?

Ans $= P(01) + P(10) = |\langle 01|\psi\rangle|^2 + |\langle 10|\psi\rangle|^2 = |\alpha_1|^2 + |\alpha_2|^2$

2) What is the probability for the ~~qubit~~ TWO QUBIT STATE $|\psi\rangle$ to be found in the state:

$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad P(|\beta_{01}\rangle) = \frac{|\alpha_1+\alpha_2|^2}{2}$$

3) And in the state:

$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \quad P(|\beta_{11}\rangle) = \frac{|\alpha_1-\alpha_2|^2}{2}$$

$\langle\psi|\psi\rangle = 1$

$\sum_{j=0}^{3}|\alpha_j|^2 = 1$

EXAMPLE:

$$|\psi\rangle = \frac{1}{\sqrt{3}}|01\rangle + \sqrt{\frac{2}{3}}|10\rangle$$

COMPUTATIONAL BASIS FOR 2 QUBITS:
$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

ANOTHER CHOICE OF 2-QUBIT BASIS:
$\left\{\frac{|00\rangle+|11\rangle}{\sqrt{2}}, \frac{|01\rangle+|10\rangle}{\sqrt{2}}, \frac{|01\rangle-|10\rangle}{\sqrt{2}}, \frac{|00\rangle-|11\rangle}{\sqrt{2}}\right\}$

$=|\beta_{01}\rangle$ $\quad =|\beta_{11}\rangle$

Summary

- Quantum measurement is qualitatively different. Most quantum measurements produce drastic "disturbances" in the state $|\Psi\rangle$: After the measurement $|\Psi\rangle$ "collapses" into the state you found it to be in.

- Projector: Hermitian operator satisfying $\hat{P}^2 = \hat{P}$. The projectors $\hat{P}_a = |a\rangle\langle a|$ formed from the elements of a basis $\{|a\rangle\}$ satisfy the completeness relation:

$$\sum_a \hat{P}_a = \hat{I}$$

- Projective measurement: For a particular basis $\{|a\rangle\}$ the outcome of the measurement is that $|\Psi\rangle$ is found to be (and collapses to) one of the basis elements $|a\rangle$:

$$|\Psi\rangle \;-\!\boxed{\underset{\{|a\rangle\}}{\mathbf{M}}}\!\rightarrow |a\rangle \text{ with prob. } |\langle a|\Psi\rangle|^2$$