Measurement of observables and the Bloch sphere

Projective measurements

- A projective measurement of state $|\Psi\rangle$ in basis $\{|a\rangle\}$ detects whether the qubit is in one of the members of the basis:

$$p(a) = |\langle a|\Psi\rangle|^2$$

$$= \mathrm{Tr}\left\{\hat{P}_a\,\hat{\rho}\right\}$$

"DENSITY MATRIX"

$|a\rangle\langle a|$ $|\psi\rangle\langle\psi|$

- Say the outcome of the measurement is $|a\rangle \in \{|a\rangle\}$. In this case the qubit collapses to the state $|a\rangle$!

$$|\Psi\rangle \xrightarrow{\;\mathbf{M}\ \{|a\rangle\}\;} |a\rangle \text{ with prob. } |\langle a|\Psi\rangle|^2$$

Why would we want to measure in a different basis?

- Consider $|+\rangle$ and $|-\rangle$. Can we distinguish them by measuring in the computational basis?

$$\begin{cases} |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{cases}$$

NOTE $\langle +|-\rangle = 0$!

MEASURE IN COMP. BASIS $\{|0\rangle, |1\rangle\}$

$$\begin{cases} p(0) = |\langle 0|\pm\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \\ p(1) = |\langle 1|\pm\rangle|^2 = \left|\pm\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \end{cases}$$

$\Rightarrow$ WE CAN NOT DISTINGUISH THEM BY MEASURING IN THE $\{|0\rangle, |1\rangle\}$ BASIS!

- But if we measure in the $\{|+\rangle, |-\rangle\}$ basis we will get either only + or only -, i.e. we are able to distinguish them.

FOR $|+\rangle$: $\begin{cases} p(+) = |\langle +|+\rangle|^2 = 1 \\ p(-) = |\langle -|+\rangle|^2 = 0 \end{cases}$ FOR $|-\rangle$: $\begin{cases} p(+) = |\langle +|-\rangle|^2 = 0 \\ p(-) = |\langle -|-\rangle|^2 = 1 \end{cases}$

Another example

- Consider

$$|+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$$

$$|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$$

- Measure in the computational basis: $\begin{cases} p(0) = |\langle 0|\pm i\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} \\ p(1) = |\langle 1|\pm i\rangle|^2 = \left|\frac{\pm i}{\sqrt{2}}\right|^2 = \frac{1}{2} \end{cases}$

- Measure in the $\{|+\rangle, |-\rangle\}$ basis: $\begin{cases} p(+) = |\langle +|\pm i\rangle|^2 = \left|\frac{1\pm i}{2}\right|^2 = \frac{(1\pm i)(1\mp i)}{4} = \frac{1+1}{4} = \frac{1}{2} \\ p(-) = |\langle -|\pm i\rangle|^2 = \left|\frac{1\mp i}{2}\right|^2 = \frac{2}{4} = \frac{1}{2} \end{cases}$

CALCULATE.

$$\langle +|\pm i\rangle = \frac{1}{\sqrt{2}}(1\ \ 1)\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm i\end{pmatrix} = \frac{1}{2}(1 \pm i)$$

- These states are indistinguishable in both basis! That's strange since $|\pm i\rangle$ look so similar to $|\pm\rangle$. The difference is the relative phase between $|0\rangle$ and $|1\rangle$. It has a large effect on the state!

Global vs. relative phase

- Changing the *global* phase

$$|\Psi\rangle \rightarrow e^{i\phi}|\Psi\rangle$$

does not change the measurement statistics. For all practical purposes the state is the same.

$$P(x) = |\langle x|e^{i\phi}|+\rangle|^2 = \underbrace{|e^{i\phi}|^2}_{=1}|\langle x|+\rangle|^2 = |\langle x|+\rangle|^2$$

Same!

- But changing the relative phase

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\Psi\rangle = \alpha|0\rangle + e^{i\phi}\beta|1\rangle$$

can have huge impact on measurement, depending on the basis you are using to measure.

Measuring an observable $\hat{A}$ = Projective measurement on set of eigenvectors of $\hat{A}$

- Consider an observable $\hat{A}$. It is Hermitian, so it becomes diagonal in the basis of eigenvectors $\{|a\rangle\}$:

$$\hat{A} = \sum_a a|a\rangle\langle a| \Leftrightarrow \hat{A}|a\rangle = a|a\rangle$$

- If the qubit is in state $|\Psi\rangle$, the probability for getting the outcome "a" is

$$p_a = |\langle a|\Psi\rangle|^2$$

- Example: SG-z measures observable $\hat{Z}$. SG-x measures $\hat{X}$. What are the probabilities?

$|+\rangle$, MEASURE SG-x $\hat{=}\ \hat{X}$ WHAT DO YOU GET? YOU GET +1 OR −1 (EIGENVALUES OF $\hat{X}$!)

IF I GET +1, WHICH STATE DID I COLLAPSE TO? ANS. $|+\rangle$!

$\hat{X}|+\rangle = (+1)|+\rangle$

Expectation value for measurements of an observable

- Say we do several measurements of observable $\hat{A}$. Each time we get one of its eigenvalues, the real number "a". What is the expectation value (average) for our measurement?

$$\langle\hat{A}\rangle \equiv \sum_a a\,p(a) = \langle\Psi|\hat{A}|\Psi\rangle$$

PROOF: $\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\left(\sum_a a|a\rangle\langle a|\right)|\psi\rangle = \sum_a a\underbrace{\langle\psi|a\rangle\langle a|\psi\rangle}_{=|\langle a|\psi\rangle|^2 = p(a)} = \sum_a a\,p(a) = \langle\hat{A}\rangle$

- Another way: $\langle\hat{A}\rangle = \mathrm{Tr}\left\{\hat{A}\,\hat{\rho}\right\}$ $\leftarrow$ SHOW THIS! MAYBE GOOD EXAM QUESTION? ☺

$|\psi\rangle\langle\psi|$

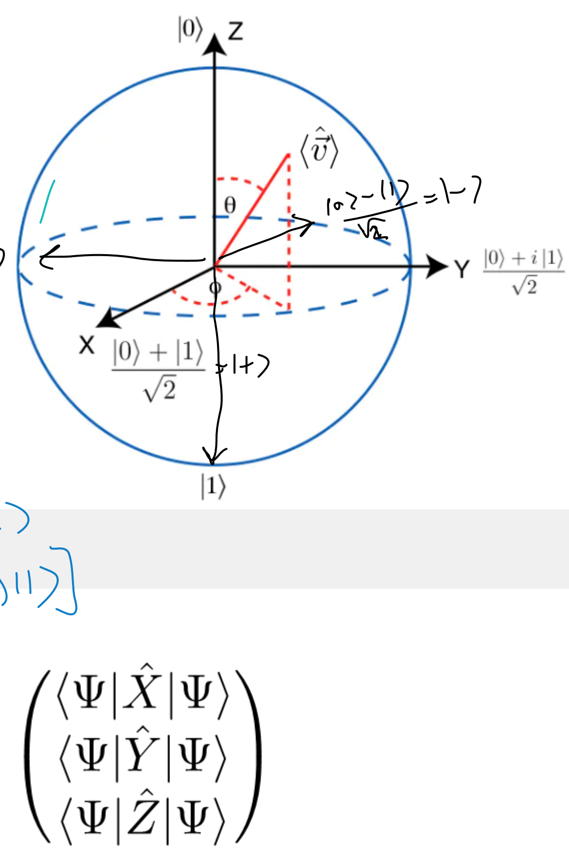
EXAMPLE. $|\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$

$$\langle\psi|\hat{X}|\psi\rangle = (\alpha^*\ \beta^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = (\alpha^*\ \beta^*)\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \alpha^*\beta + \beta^*\alpha$$

The Bloch sphere: Geometric representation for one qubit

- For a given qubit state $|\Psi\rangle$, consider the vector for the expectation values for the Pauli operators (average spin direction):

$$\langle\hat{\vec{v}}\rangle = \begin{pmatrix}\langle\psi|\hat{x}|\psi\rangle\\ \langle\psi|\hat{y}|\psi\rangle\\ \langle\psi|\hat{z}|\psi\rangle\end{pmatrix}$$



$$\boxed{|\psi\rangle = \cos\left(\tfrac{\theta}{2}\right)|0\rangle + \sin\left(\tfrac{\theta}{2}\right)e^{i\phi}|1\rangle}$$

NOTE GLOBAL PHASE CAN ALWAYS BE REMOVED:

E.G. $|\psi\rangle = e^{i\gamma}\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\gamma'}\sin\left(\frac{\theta}{2}\right)|1\rangle$

$= e^{i\gamma}\left[\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i(\gamma'-\gamma)}\sin\left(\frac{\theta}{2}\right)|1\rangle\right]$

GLOBAL PHASE, CAN DROP!

NOTE: FOR $|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|1\rangle$ CAN SHOW THAT $|\langle\hat{\vec{v}}\rangle| = 1$ SO STATE $|\psi\rangle$ IS ALWAYS AT SURFACE WITH $|\langle\hat{\vec{v}}\rangle| = 1$ OF BLOCH SPHERE

$$\langle 1|\hat{X}|1\rangle = (0\ 1)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = (0\ 1)\begin{pmatrix}1\\ 0\end{pmatrix} = 0$$

Let's get to work…

- Fill up the table by calculating $\langle\hat{\vec{v}}\rangle = \begin{pmatrix}\langle\Psi|\hat{X}|\Psi\rangle\\ \langle\Psi|\hat{Y}|\Psi\rangle\\ \langle\Psi|\hat{Z}|\Psi\rangle\end{pmatrix}$

$\|\Psi\rangle$	$\langle\hat{\vec{v}}\rangle$
$\|0\rangle$	
$\|1\rangle$	$-\hat{z}$
$\|+\rangle$	
$\|-\rangle$	
$\|+i\rangle$	
$\|-i\rangle$	

Summary

- Projective measurement: $|\Psi\rangle \xrightarrow{\;\mathbf{M}\ \{|a\rangle\}\;} |a\rangle \text{ with prob. } |\langle a|\Psi\rangle|^2$

- The outcome of a measurement of an observable Â is one of its eigenvalues with probability $=|\langle a|\Psi\rangle|^2$ where |a> is the eigenvector associated to the eigenvalue.

- Expectation value for an observable: $\langle\hat{A}\rangle = \sum_a a\,p(a) = \langle\Psi|\hat{A}|\Psi\rangle$

- For a given one-qubit state $|\Psi\rangle$, the vector $\langle\vec{v}\rangle = \begin{pmatrix}\langle\hat{X}\rangle\\ \langle\hat{Y}\rangle\\ \langle\hat{Z}\rangle\end{pmatrix}$ gives a geometrical representation. It's called Bloch sphere.