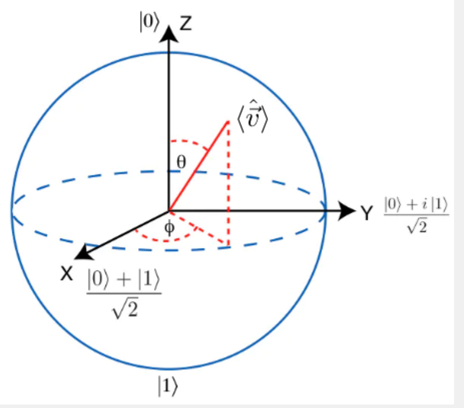
Unitary operators and quantum circuits

Let's get to work…

- Fill up the table by calculating $\langle\hat{\vec{v}}\rangle = \begin{pmatrix} \langle\Psi|\hat{X}|\Psi\rangle \\ \langle\Psi|\hat{Y}|\Psi\rangle \\ \langle\Psi|\hat{Z}|\Psi\rangle \end{pmatrix}$

$\lvert\Psi\rangle$	$\langle\hat{\vec{v}}\rangle$
$\lvert 0\rangle$	$+\hat{z}$
$\lvert 1\rangle$	$-\hat{z}$
$\lvert +\rangle$	$+\hat{x}$
$\lvert -\rangle$	$-\hat{x}$
$\lvert +i\rangle$	$+\hat{y}$
$\lvert -i\rangle$	$-\hat{y}$



Unitary operators (norm-preserving)

- A important class of operators satisfy the *unitary property*:

$$\boxed{\hat{U}^\dagger = \hat{U}^{-1}} \qquad \hat{U}^{-1}\cdot\hat{U} = \hat{I}, \quad \hat{U}\cdot\hat{U}^{-1} = \hat{I}$$

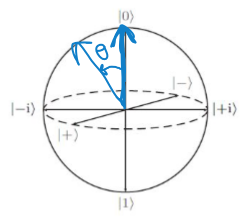
- When applied to a state, they preserve its normalization:

$$|\varphi\rangle = \hat{U}|\psi\rangle \Rightarrow \langle\varphi|\varphi\rangle = (\hat{U}|\psi\rangle)^\dagger\cdot(\hat{U}|\psi\rangle) = \langle\psi|\underbrace{\hat{U}^\dagger\cdot\hat{U}}_{=\hat{I}}|\psi\rangle = \langle\psi|\psi\rangle$$

- All "quantum gates" preserve norm, so they are all unitary. Example:

ROTATION BY ANGLE θ ABOUT AXIS $\hat{x}$

$$\hat{R}_x(\theta) = \mathrm{e}^{-i\frac{\theta}{2}\hat{X}} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$



NOTE: $\hat{R}_x^\dagger(\theta) = e^{+i\frac{\theta}{2}\hat{X}^\dagger} = e^{+i\frac{\theta}{2}\hat{X}} = \begin{pmatrix} \cos(\frac{\theta}{2}) & +i\sin(\frac{\theta}{2}) \\ +i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} \neq \hat{R}_x(\theta)$

NOT HERMITIAN! (BUT IT IS UNITARY, CHECK!)

Unitary operators describe basis changes

- Hadamard operator $\hat{H}$ takes computational basis into $\{|\pm\rangle\}$basis:

$$\hat{H}|0\rangle = |+\rangle$$
$$\hat{H}|1\rangle = |-\rangle$$

$$\hat{H} = \begin{pmatrix} \langle 0|\hat{H}|0\rangle & \langle 0|\hat{H}|1\rangle \\ \langle 1|\hat{H}|0\rangle & \langle 1|\hat{H}|1\rangle \end{pmatrix} = \begin{pmatrix} \langle 0|+\rangle & \langle 0|-\rangle \\ \langle 1|+\rangle & \langle 1|-\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$\hat{H} = \hat{H}^\dagger$ (HADAMARD IS HERMITIAN)

$\hat{H}^\dagger\hat{H} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \hat{I}$ (HADAMARD IS UNITARY!)

NOTE: SAME FOR PAULI OPERATORS! THEY ARE HERMITIAN AND UNITARY!

- General proof that any unitary Û maps one basis into another

$$\{|\phi_1\rangle, |\phi_2\rangle\} \xrightarrow{\hat{U}} \{\underbrace{|\psi_1\rangle = \hat{U}|\phi_1\rangle, |\psi_2\rangle = \hat{U}|\phi_2\rangle}_{\text{NEED TO PROVE THAT THIS IS A BASIS!}}\}$$

ORTHONORMALITY: $\langle\psi_i|\psi_j\rangle = \langle\phi_i|\underbrace{\hat{U}^\dagger\hat{U}}_{=\hat{I}}|\phi_j\rangle = \langle\phi_i|\phi_j\rangle = \delta_{ij}$ ✓

COMPLETENESS: $|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2| = \hat{U}[\underbrace{|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|}_{=\hat{I}}]\hat{U}^\dagger = \hat{U}\hat{U}^\dagger = \hat{I}$ ✓

Unitary operators describe rotations in the Bloch sphere

- Rotation operator about angle θ and axis $\hat{n}$:

$$\hat{R}_{\hat{n}}(\theta) = \mathrm{e}^{-i\frac{\theta}{2}\hat{\sigma}_n} = \cos\left(\frac{\theta}{2}\right)\hat{I} - i\sin\left(\frac{\theta}{2}\right)\hat{\sigma}_n$$

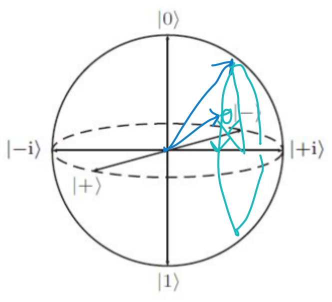
$\hat{\sigma}_n = n_x\hat{X} + n_y\hat{Y} + n_z\hat{Z}$

- Example: Rotation about y axis:

$$\hat{R}_y(\theta) = e^{-i\frac{\theta}{2}\hat{Y}} = \cos(\tfrac{\theta}{2})\hat{I} - i\sin(\tfrac{\theta}{2})\hat{Y}$$

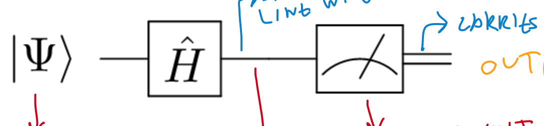
$$= \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ +\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$$

ROTATION IN "SU(2)" SPECIAL UNITARY
TECHNICAL NAME FOR THE 1 QUBIT SPACE.



Quantum circuits

- Quantum circuits give a nice representation of the action of unitaries or gates. We read the circuit from left to right and the measurement is always on the computational basis:



SINGLE LINE WIRE = CARRIES QUANTUM STATE

CARRIES CLASSICAL BIT, $x = 0, 1$

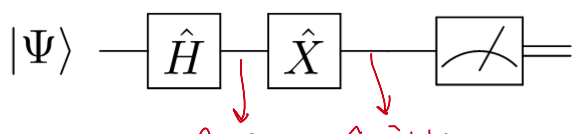
OUTPUT: $\begin{cases} x = 0 \text{ WITH } p(0) = |\langle 0|\hat{H}|\psi\rangle|^2 \\ x = 1 \text{ WITH } p(1) = |\langle 1|\hat{H}|\psi\rangle|^2 \end{cases}$

INPUT STATE

$\hat{H}|\psi\rangle$

MEASUREMENT IN $\{|0\rangle, |1\rangle\}$ BASIS

- Because we read from left to right the action of two operators is "reversed":



$\hat{H}|\psi\rangle$

$\hat{X}\hat{H}|\psi\rangle$

REVERSE OF CIRCUIT!

Summary

- Quantum gates are described by unitary operators (also known as norm-preserving):

$$\hat{U}^\dagger = \hat{U}^{-1}$$

- Unitary operators provide a map that links two different basis. They can be visualized as rotations of states represented by the vector $\langle\vec{v}\rangle$ in the Bloch sphere.
- Writing down all gates in matrix form is tedious, so we use quantum circuits:

What does this circuit do?