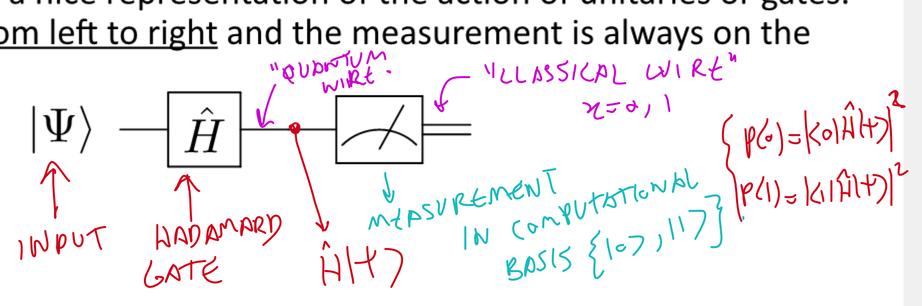
Quantum circuits for 1 and 2 qubits

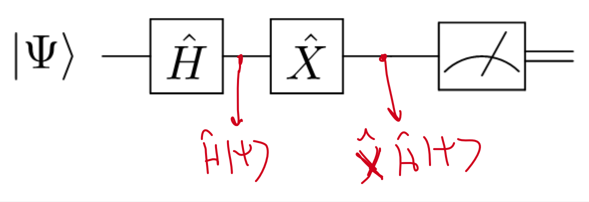
Quantum circuits

- Quantum circuits give a nice representation of the action of unitaries or gates. We read the circuit from left to right and the measurement is always on the computational basis:



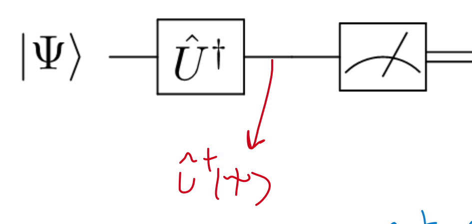
$\begin{cases} P(0) = |\langle 0|\hat{H}|\psi\rangle|^2 \\ P(1) = |\langle 1|\hat{H}|\psi\rangle|^2 \end{cases}$

- Because we read from left to right the action of two operators is "reversed":



Quantum measurement in a different basis

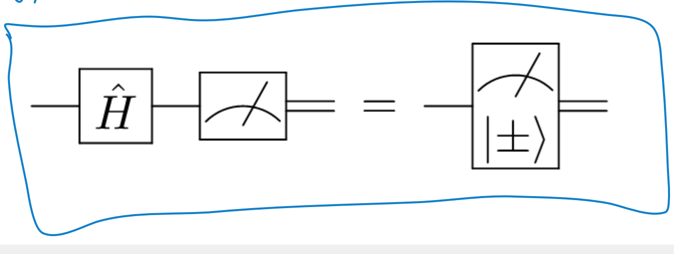
- This circuit does quantum measurement in the basis $\{\underbrace{\hat{U}|0\rangle}_{=|0'\rangle}, \underbrace{\hat{U}|1\rangle}_{=|1'\rangle}\}$!



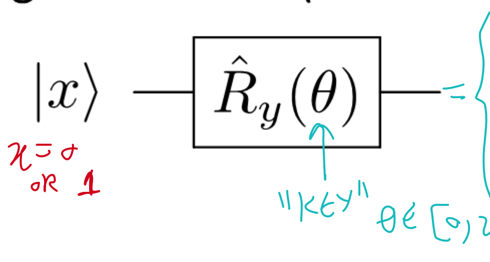
$\begin{cases} P(0) = |\langle 0|\hat{U}^\dagger|\psi\rangle|^2 = |(\hat{U}|0\rangle)^\dagger|\psi\rangle|^2 = P(\hat{U}|0\rangle = |0'\rangle) \\ P(1) = |\underbrace{\langle 1|\hat{U}^\dagger}_{=(\hat{U}|1\rangle)^\dagger}|\psi\rangle|^2 = P(\hat{U}|1\rangle) \end{cases}$

EXAMPLE: FOR $\hat{U} = \hat{H}$, $\hat{U}^\dagger = \hat{H}^\dagger = \hat{H}$

- For example,



Circuit for keeping a bit secret (with "one-try eavesdrop")

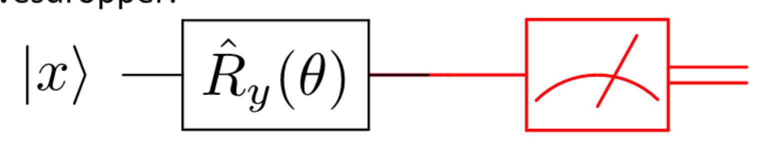


$= \begin{cases} \text{IF } x = 0 & \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) \end{pmatrix} \\ \text{IF } x = 1 & \begin{pmatrix} -\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{pmatrix} \end{cases}$

$\hat{R}_y(\theta) = e^{-i\frac{\theta}{2}\hat{y}} = \cos(\frac{\theta}{2}) - i\sin(\frac{\theta}{2})\hat{y}$

$= \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix}$

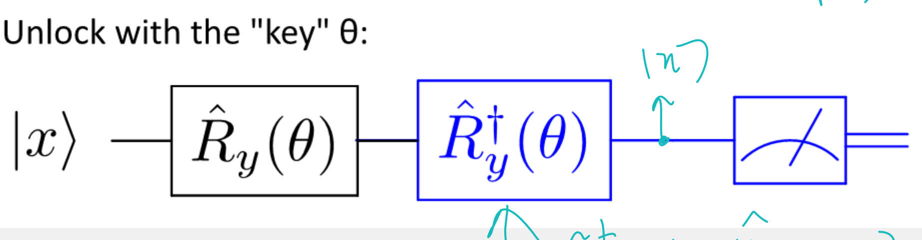
- Eavesdropper:



$P(0) = \cos^2(\frac{\theta}{2})$ OR $\sin^2(\frac{\theta}{2})$

$P(1) = \sin^2(\frac{\theta}{2})$ OR $\cos^2(\frac{\theta}{2})$

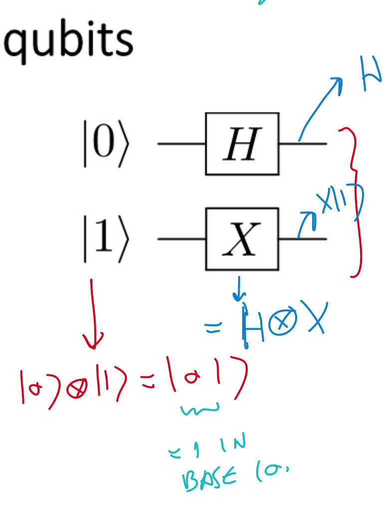
- Unlock with the "key" θ:



$\begin{cases} P(x) = 1 \\ P(\bar{x}) = 0 \end{cases}$ $(\bar{x} = \text{NOT}(x))$

$\hat{R}_y^\dagger(\theta) = \hat{R}_y(-\theta)$

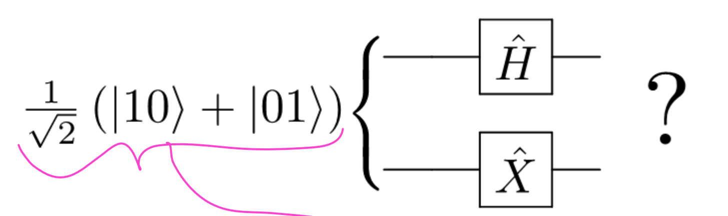
Circuit for two qubits



$= H \otimes X\, |0\rangle|1\rangle$
$= (H|0\rangle) \otimes (X|1\rangle)$
$= (H|0\rangle)(X|1\rangle)$

$|01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad H\otimes X = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & -1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{array}\right)$

Let's see if we got this:



$|?\rangle = \frac{1}{\sqrt{2}}\left(\begin{array}{cc|cc} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{array}\right)\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$

A challenge for you...

- Consider the 2-qubit state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- Find

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \text{ and } |\phi\rangle = \gamma|0\rangle + \delta|1\rangle$$

So that

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle$$

THIS IS IMPOSSIBLE!
$\alpha, \beta, \gamma, \delta$ DO NOT EXIST! (SORRY ..)

Entangled states

- The state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is **entangled**.

DEF. $|\Psi\rangle$ IS ENTANGLED IF IT CAN NOT BE WRITTEN AS $|\Psi\rangle = |\psi\rangle|\varphi\rangle$.

- We cannot describe the two qubits individually, we can only describe their combined state.

Paraphrasing John Preskill: *it's like you're reading a book, but instead of reading the pages sequentially, you have to read it all at the same time in order to understand it.*

Entanglement: Qubits are "correlated"

- Furthermore, the measurement outcomes of $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ are perfectly correlated.

- For example, if you measure the 1st qubit and get 0, you'll get 0 for the 2nd qubit as well!
- Entanglement is not limited to two qubits. In principle we can entangle as many as we like:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\overbrace{0000\cdots}^{n \text{ QUBITS}}\rangle + |1111\cdots\rangle)$$

"CAT STATES"

- A measurement outcome of 0 on the 1st qubit means you'll get 0 in all other qubits as well.

OTHER EXAMPLES.

$|\psi\rangle = \frac{1}{\sqrt{2}}(|001\rangle + |100\rangle)$

$|\psi'\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)|0\rangle$

How to make entangled states?

- All the gates we discussed so far are 1-qubit gates – they ~~generate~~ TAKE NON-entangled states ~~from~~ INTO non-entangled ones!

$$U_1 \otimes U_2|00\rangle = \underbrace{U_1|0\rangle \otimes U_2|0\rangle}_{\text{PRODUCT STATE}}$$

PRODUCT STATE ($U_1\otimes U_2$ DOES NOT ENTANGLE!)

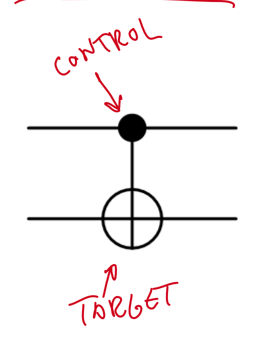
- We need a new kind of gate: The 2-qubit gate. The most popular choice is CNOT= "control-not":

CONTROL TARGET

$$\text{CNOT}|00\rangle = |00\rangle$$
$$\text{CNOT}|01\rangle = |01\rangle$$
$$\text{CNOT}|10\rangle = |11\rangle$$
$$\text{CNOT}|11\rangle = |10\rangle$$

TARGET FLIPPED!

TARGET FLIPPED!

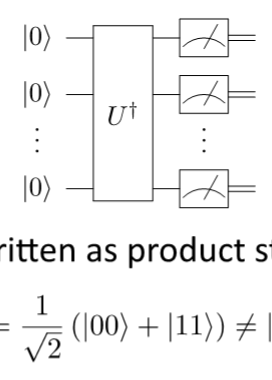


$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{=|10\rangle} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{=|11\rangle}$

Summary

- Writing down all gates in matrix form is tedious, so we use quantum circuits:

What does this circuit do?



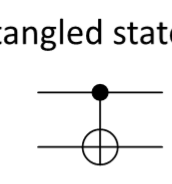
MEASURES IN ANOTHER n-QUBIT BASIS!
$\{U|0\ldots0\rangle, U|0\ldots1\rangle, \ldots\}$

- Entangled states can not be written as product states:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \neq |\phi\rangle \otimes |\psi\rangle$$

- Two-qubit gates are needed to create entangled states. Usual choice is CNOT:

$$\text{CNOT}|00\rangle = |00\rangle$$
$$\text{CNOT}|01\rangle = |01\rangle$$
$$\text{CNOT}|10\rangle = |11\rangle$$
$$\text{CNOT}|11\rangle = |10\rangle$$



$\left(\text{CNOT}\ \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle$

HERE I USED CNOT TO DISENTANGLE!