Two-qubit circuits

NEW NOTATION:

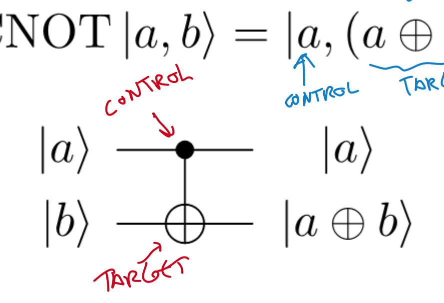
1) NO HAT FOR OPERATORS: $\hat{A} \to A$
2) ARROW FOR VECTORS: $\langle \vec{n} \rangle$ (NO BOLD)
HAT FOR UNIT VECTORS, $\hat{n}$
$\langle \vec{n} \rangle$ OR $\langle \hat{n} \rangle$

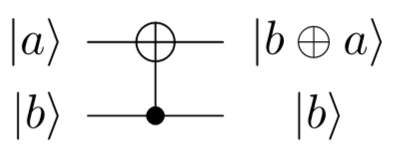
CNOT "control-not" gate:

MODULO 2 ADDITION

$$\text{CNOT}\,|a,b\rangle = |a,(a \oplus b)\rangle$$

CONTROL TARGET





$a, b = 0, 1$

$|00\rangle \xrightarrow{\text{CNOT}} |0, 0\oplus 0\rangle = |00\rangle$

$|10\rangle \to |1, 1\oplus 0\rangle = |11\rangle$

⋮

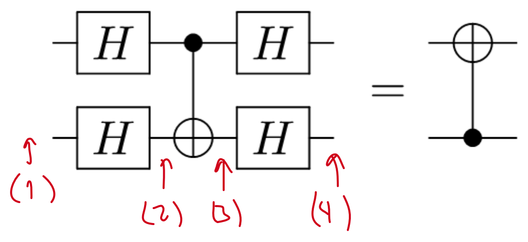
CTR TARGET

$\left\{ \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \dots \right\}$

$\downarrow$ CNOT

$\frac{|00\rangle + |10\rangle}{\sqrt{2}}$

Prove:



$\begin{cases} H|0\rangle = |+\rangle \\ H|1\rangle = |-\rangle \end{cases}$

(1) (2) (3) (4)

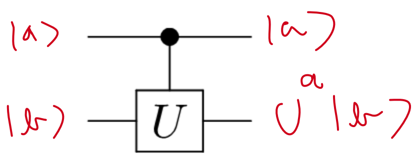
$\left.\begin{cases} |0\rangle \to |+\rangle \\ |0\rangle \to |+\rangle \end{cases}\right\} = |++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \xrightarrow{\text{CNOT}} |++\rangle \longrightarrow |00\rangle$

$|01\rangle \to \underbrace{|+-\rangle}_{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)} = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle) \to \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) = |-\rangle|-\rangle \to |11\rangle$

$|10\rangle \to |-+\rangle = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle) \to |-+\rangle \longrightarrow |10\rangle$

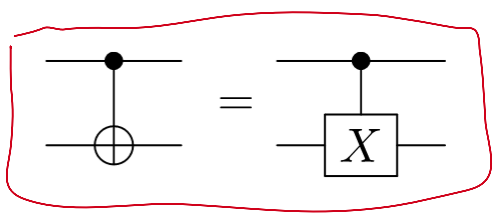
$|11\rangle \to |--\rangle = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle) \to |+-\rangle \longrightarrow |01\rangle$

Control-U operation

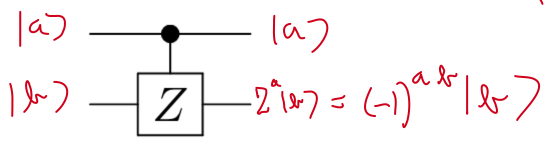


$\begin{cases} CU|00\rangle = |0\rangle|0\rangle \\ CU|01\rangle = |0\rangle|1\rangle \\ CU|10\rangle = |1\rangle U|0\rangle \\ CU|11\rangle = |1\rangle U|1\rangle \end{cases}$

- CNOT=control-X



- CZ=control-Z is another popular two-qubit gate:



$\boxed{CZ = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -1 \end{pmatrix}}$

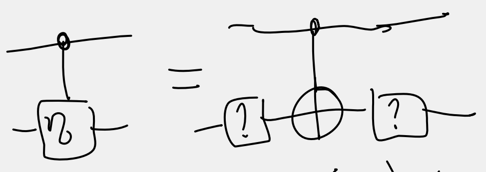
$CZ|11\rangle = -|11\rangle$

$CZ\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right) = \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)$

$\left.\begin{matrix} Z|0\rangle = |0\rangle \\ Z|1\rangle = -|1\rangle \end{matrix}\right\} Z|a\rangle = (-1)^a|a\rangle$

$\cancel{Z|01\rangle} =$

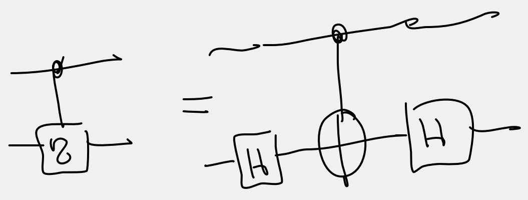
$\underbrace{(Z \otimes I)}_{Z_1}|01\rangle = (Z|0\rangle) \otimes (I|1\rangle) = |01\rangle$



$HZH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$

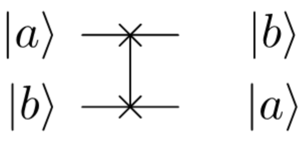
$\Rightarrow$



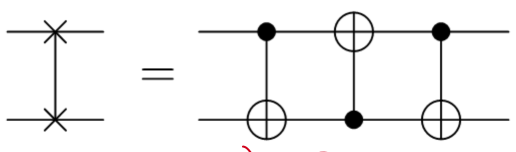
Swap gate

$$\text{SWAP}\,|\psi\rangle\,|\phi\rangle = |\phi\rangle\,|\psi\rangle$$

- "Swaps" the state of two kets:



- Here is how to "compile" a swap gate using CNOTs :

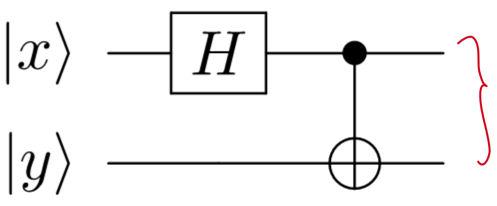


3 GATES HERE

- We say that SWAP has quantum depth = 3.

Time to work…

- Fill up the table



$|\beta_{xy}\rangle$ "BELL STATES"

$|\beta_{xy}\rangle = \frac{1}{\sqrt{2}}\left(|0\,y\rangle + (-1)^x|1\,\bar{y}\rangle\right)$

x	y	$\vert\text{output}\rangle$
0	0	$\vert\beta_{00}\rangle = \frac{1}{\sqrt{2}}(\vert 00\rangle + \vert 11\rangle)$
0	1	$\vert\beta_{01}\rangle = \frac{1}{\sqrt{2}}(\vert 01\rangle + \vert 10\rangle)$
1	0	$\vert\beta_{10}\rangle = \frac{1}{\sqrt{2}}(\vert 00\rangle - \vert 11\rangle)$
1	1	$\vert\beta_{11}\rangle = \frac{1}{\sqrt{2}}(\vert 01\rangle - \vert 10\rangle)$

Summary

- Two-qubit gates: CNOT, CZ, Control-U, SWAP. They are often related to each other, e.g. :