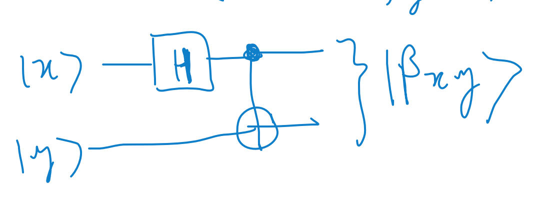
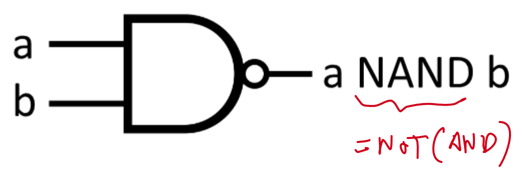
Universal set of quantum gates

BELL BASIS $\{|\beta_{xy}\rangle\}_{x,y=0,1}$



"Classical conventional" computers: Reversibility and universality

- Classical digital computers are made of Boolean gates:



a	b	NAND
0	0	1
0	1	1
1	0	1
1	1	0

- They are not reversible.

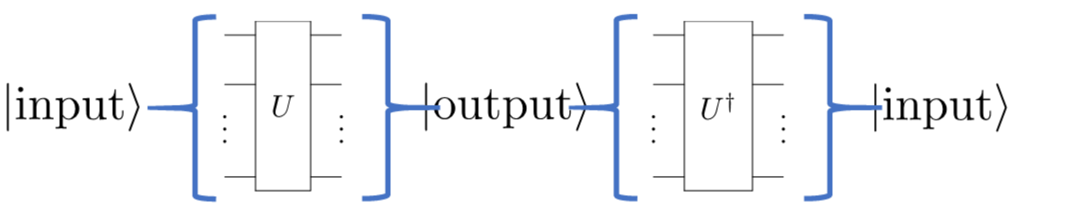
IF YOU TELL ME THAT THE OUTPUT IS 1, I CAN'T TELL WHAT IS THE INPUT! 3 OPTIONS...

- Any Boolean function f(x)=y where x $\in\{0,1\}^n$ (n-BIT STRING) and y $\in\{0,1\}^m$ (m-BIT STRING) can be computed with a series of NAND gates (plus wires and FANOUT, the ability to copy bits).

$\boxed{\text{NAND is universal}}$

What about quantum gates?

- They are always reversible (unless you measure!): $U^\dagger U = I$!

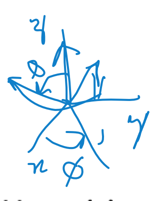


- \# of qubits in input = # qubits in output.
- Universality in QC = A set of gates that can "compile" any imaginable unitary transformation taking a n-qubit state into another n-qubit state.
- Quantum universality? How many/which kinds of gates do we need?

Back to 1-qubit gates: Family of all qubit unitaries?

- How about:

$R_z(\phi)R_y(\theta)$



$\theta \in [0,\pi]$
$\phi \in [0,2\pi)$

- Not sufficient! These are non-Hermitian, so they can't describe H, X, Y, Z.

$$H = iR_y\left(\frac{\pi}{2}\right)R_z(\pi) = i\begin{pmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}\begin{pmatrix}e^{-i\frac{\pi}{2}} & 0\\ 0 & e^{i\frac{\pi}{2}}\end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} = \frac{i}{\sqrt{2}}\begin{pmatrix}-i & -i\\ -i & i\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H$$

- Here two ways to parametrize the whole family:

$$U = e^{i\alpha}R_{\hat{n}}(\theta) \quad \text{or} \quad U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta)$$

CONTINUOUS PARAMETERS! (1, 2, 1) — FOUR PARAMETERS: $\alpha, \beta, \gamma, \delta$!

- The set $\{R_z(\delta), R_y(\delta), e^{i\alpha}I\}$ is said to form a **"universal"** set of 1-qubit U's.

A simpler universal set for 1-qubit unitaries

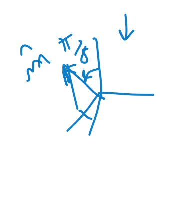
- It is possible to prove that H and T gates can be used to approximate any 1-qubit unitary to arbitrary accuracy.

"$\pi/8$ gate" $-\boxed{T}-$ $\begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix} = e^{i\frac{\pi}{8}}\begin{pmatrix}e^{-i\frac{\pi}{8}} & 0\\ 0 & e^{i\frac{\pi}{8}}\end{pmatrix} = e^{i\frac{\pi}{8}}R_z\left(\frac{\pi}{4}\right)$

YOU CAN PROVE THAT

$$\boxed{R_{\hat{n}}(\theta) = (HTH)(THT)(HTH)\cdots T}$$

$THTH = R_{\hat{m}}(\cos^2(\frac{\pi}{8}))$



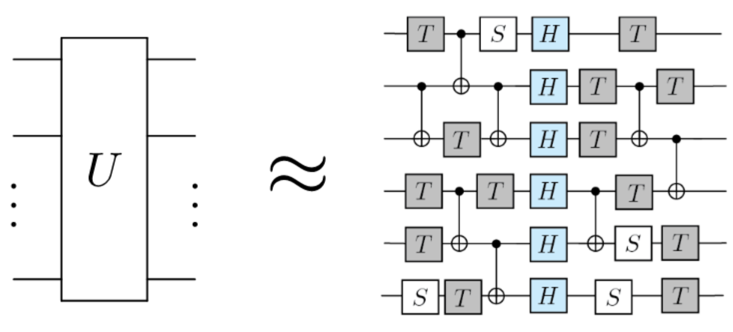
- The S gate is added to provide fault-tolerance:

"phase gate" $-\boxed{S}-$ $\begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$

The standard set of universal gates

- H, T, S and CNOT between two nearest neighbour qubits provide a universal set for ALL multi-qubit unitaries!

$$\{H, T, S, CNOT\}$$



- A "quantum compiler" is needed to translate a unitary gate into a product of "native gates" implemented by the quantum hardware.

Programming a QC to evaluate Boolean f(x)

- One bit case: $f : \{0,1\} \to \{0,1\}$
- Try $f_0(x)=0$: $|x\rangle \to \boxed{U_{f_0}} \to |f_0(x)\rangle = |0\rangle$

IMPOSSIBLE WITH UNITARY U_{f_0}

NOT REVERSIBLE! IT CAN NOT BE DONE...

SUPPOSE $U_{f_0} = |0\rangle(\langle 0| + \langle 1|)$ $\left\{\begin{array}{l} U_{f_0}|0\rangle = |0\rangle \\ U_{f_0}|1\rangle = |0\rangle \end{array}\right.$ WORKS FOR COMP. BASIS.

BUT: $U_{f_0}(\alpha|0\rangle + \beta|1\rangle) = |0\rangle\underbrace{(\alpha+\beta)}_{\neq 1}$ NOT NORMALIZED! $\Rightarrow U_{f_0}$ IS NOT UNITARY...

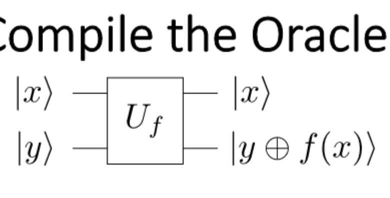
- It's impossible with only 1 qubit. Need 2 qubits, otherwise U_f is not reversible. Here is how:

U_f IS CALLED "BIT ORACLE"

$$U_f(|x\rangle|y\rangle) = |x\rangle|y \oplus f(x)\rangle$$

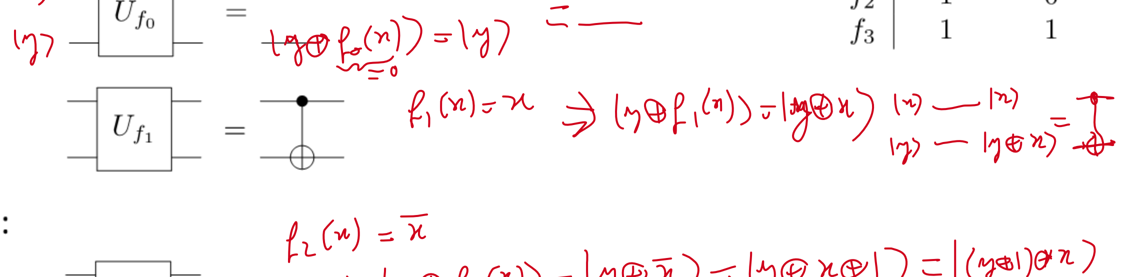
(mod 2 ADDITION)

Time to work. Compile the Oracle U_f for all 4 f's



	$x=0$	$x=1$
f_0	0	0
f_1	0	1
f_2	1	0
f_3	1	1

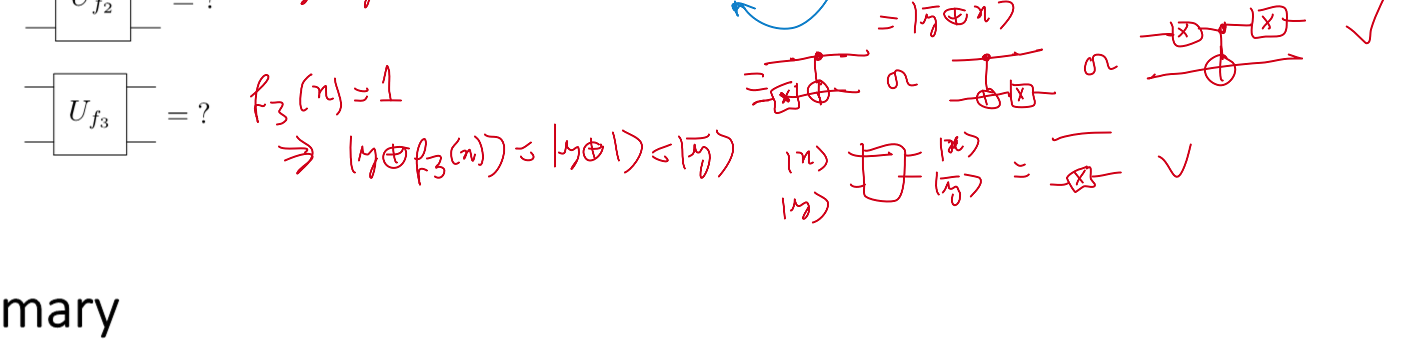
- Show:



$|y \oplus f_0(x)\rangle = |y\rangle$ ✓

$f_1(x) = x \Rightarrow |y \oplus f_1(x)\rangle = |y \oplus x\rangle$

- Find:



$f_2(x) = \bar{x}$

$\Rightarrow |y \oplus f_2(x)\rangle = |y \oplus \bar{x}\rangle = |y \oplus x \oplus 1\rangle = |(y \oplus 1) \oplus x\rangle = |\bar{y} \oplus x\rangle$ ✓

$f_3(x) = 1$

$\Rightarrow |y \oplus f_3(x)\rangle = |y \oplus 1\rangle = |\bar{y}\rangle$ ✓

Summary

- Quantum gates are always reversible and require the same number of qubits in the input and output.
- Universality: We only need a finite set of gates to approximate an arbitrary unitary operation. Standard set of universal gates:

$$\{H, T, S, CNOT\}$$

- The quantum compiler is the software layer that translates a desired unitary into native gates implemented by the quantum hardware. Some unitaries are quite costly – we say they have "large quantum depth".