No-cloning theorem, quantum teleportation, quantum error correction

No-cloning theorem

- It turns out that quantum states *can not be copied or cloned*. This is a consequence of *linearity*. Proof: Say U is a unitary operator that is able to clone arbitrary states $|\psi\rangle, |\phi\rangle$:

$$U(|\psi\rangle|0\rangle) = |\psi\rangle|\psi\rangle \text{ and } U(|\phi\rangle|0\rangle) = |\phi\rangle|\phi\rangle$$

LINEARITY: $U[(\alpha|\psi\rangle+\beta|\phi\rangle)|0\rangle] = \alpha U[|\psi\rangle|0\rangle] + \beta U[|\phi\rangle|0\rangle] = \alpha|\psi\rangle|\psi\rangle + \beta|\phi\rangle|\phi\rangle$

BUT: $= (\alpha|\psi\rangle+\beta|\phi\rangle)(\alpha|\psi\rangle+\beta|\phi\rangle) = \alpha^2|\psi\rangle|\psi\rangle + \beta^2|\phi\rangle|\phi\rangle + \alpha\beta|\psi\rangle|\phi\rangle + \beta\alpha|\phi\rangle|\psi\rangle$

$\Rightarrow$ THESE TWO ONLY AGREE WHEN EITHER α OR β ARE ZERO!

- It's also impossible to *clone approximately*.

$U(|\psi\rangle|0\rangle) \approx |\psi\rangle|\psi\rangle$ AND $U(|\phi\rangle|0\rangle) \approx |\phi\rangle|\phi\rangle$

BUT U UNITARY MUST PRESERVE INNER PRODUCT:

$\langle\psi|\langle 0||\phi\rangle|0\rangle = \langle\psi|\langle\psi||\phi\rangle|\phi\rangle$

$\boxed{\langle\psi|\phi\rangle = \langle\psi|\phi\rangle^2}$ THIS APPROX. WOULD ONLY HOLD FOR $\langle\psi|\phi\rangle \simeq 0$ OR $\langle\psi|\phi\rangle \simeq 1$.

Algorithm for quantum teleportation

- While |ψ> can not be cloned, Alice can *teleport* (reassign to another qubit) her state to Bob without corrupting it. The price she pays for this is that her qubit (originally |ψ>) is reset to |0> or |1>.
- This can be done even when Bob is far away from her. All it takes is that Alice is able to send classical information to Bob (e.g. a phone call) and crucially that *they share an entangled state*. Alice (a) and Bob (b) start with the state:

$$|\psi\rangle_a|\beta_{00}\rangle_{ab} = \underbrace{(\alpha|0\rangle_a+\beta|1\rangle_a)}_{\text{A WANTS TO TELEPORT}}\underbrace{\frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_b+|1\rangle_a|1\rangle_b)}_{\text{SHARED ENTANGLED PAIR.}}$$

$|\psi\rangle$ — 2, $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ — 1, 0

- Alice applies a CNOT with control on her $|\psi\rangle_a$ qubit and target on her member of the entangled pair. She gets:

$CNOT_{21}\left(|\psi\rangle\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\right) = \alpha|0\rangle_a\frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_b+|1\rangle_a|1\rangle_b) + \beta|1\rangle_a\frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b+|0\rangle_a|1\rangle_b)$

- Next she applies a Hadamard H to her first qubit:

$H_2 CNOT_{21}\left(|\psi\rangle\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)\right) = \alpha\left(\frac{|0\rangle_a+|1\rangle_a}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}(|0\rangle_a|0\rangle_b+|1\rangle_a|1\rangle_b) + \beta\left(\frac{|0\rangle_a-|1\rangle_a}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}(|1\rangle_a|0\rangle_b+|0\rangle_a|1\rangle_b)$

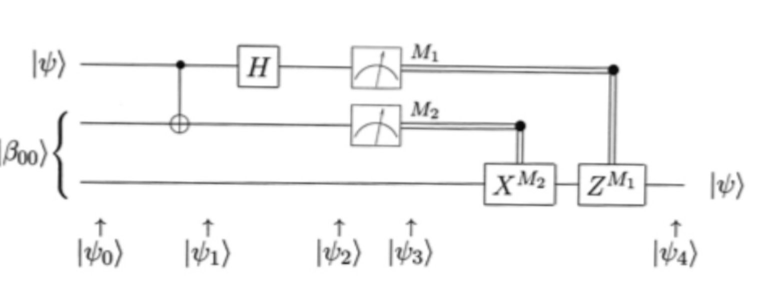
$= \frac{1}{2}|0\rangle_a|0\rangle_a(\alpha|0\rangle_b+\beta|1\rangle_b)$
$+ \frac{1}{2}|0\rangle_a|1\rangle_a(\alpha|1\rangle_b+\beta|0\rangle_b)$
$+ \frac{1}{2}|1\rangle_a|0\rangle_a(\alpha|0\rangle_b-\beta|1\rangle_b) + \frac{1}{2}|1\rangle_a|1\rangle_a(\alpha|1\rangle_b-\beta|0\rangle_b)$

- Now Alice measures both qubits in her possession. Depending on the outcome of her measurement, Bob will end up with a different state:

Alice measures	Bob gets	BOB APPLIES
00	$(\alpha\|0\rangle_b+\beta\|1\rangle_b)$	I
01	$(\alpha\|1\rangle_b+\beta\|0\rangle_b)$	X
10	$(\alpha\|0\rangle_b-\beta\|1\rangle_b)$	Z
11	$(\alpha\|1\rangle_b-\beta\|0\rangle_b)$	ZX

- Finally, Alice calls Bob on the phone and tells him her measurement outcome. If she got 00, Bob knows he has her state ☺. If she got 01, Bob applies X to his state ☺. If she got 10, Bob applies Z ☺. If she got 11, Bob applies ZX ☺. That's it, Alice state has been teleported!

Your turn: Verify that this circuit does quantum teleportation



Error correction

- Quantum computers are quite sensitive to interaction with environment (electric and magnetic noise, energy relaxation, etc). But how to correct for errors if we can not tell which state the qubit is in?
- Classical error correction: Encode using repetition or "redundancy",

$$0 \to 000$$
$$1 \to 111$$

- Suppose after some time t a single bit flip occurs with probability p<<1:

000 → 100, 000 → 010 ...
111 → 011, 111 → 101, ...

- After time t the original state can be "decoded" by majority voting. This reduces the error prob. to p^2<<p (prob. for two bit flips within t as opposed to just one).

100 → 0, 010 → 0
011 → 1, 101 → 1, ...

Can quantum error correction achieve the impossible?

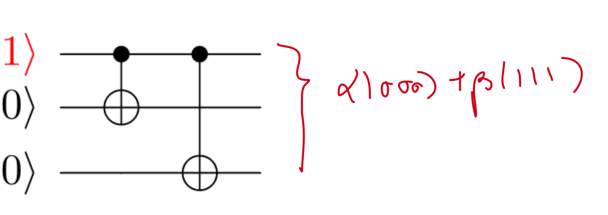
- No-cloning theorem forbids repetition encoding
- A continuum of different errors can affect a qubit (similar to analog computers: No error correction in a finite # of steps exists for them!)
- Measurement erases most of quantum information, so how can we decode by majority voting?

Entanglement provides a solution to these problems!

- Encode in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \to \boxed{\alpha|000\rangle + \beta|111\rangle}$
- Decode by measuring *correlations* without corrupting the state.

Quantum error correction: 3-qubit bit flip code

- Encode:



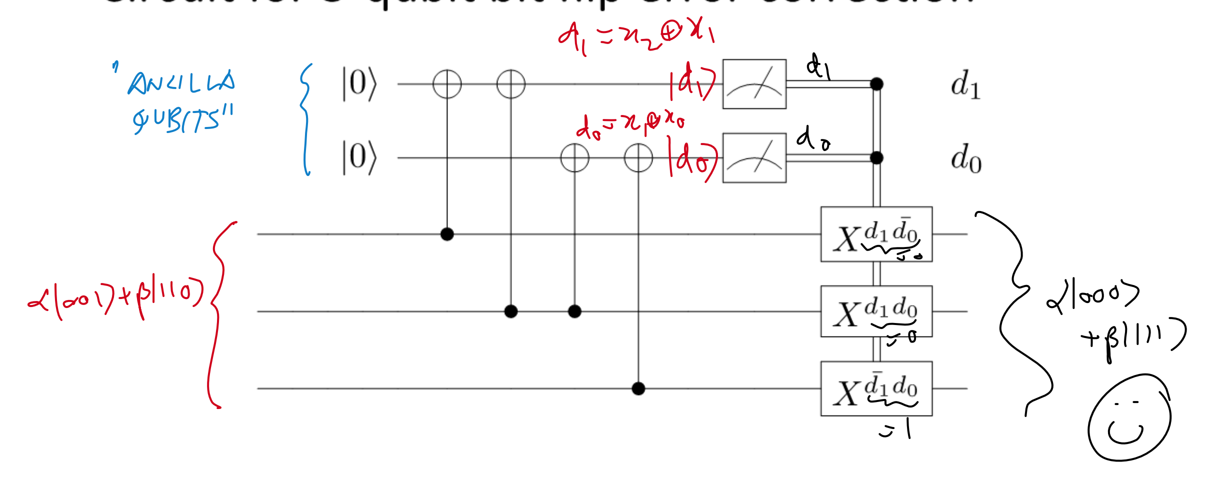
- After time t, we consider 4 possibilities:

$|\psi\rangle \to \alpha|000\rangle + \beta|111\rangle$
$\alpha|001\rangle + \beta|110\rangle$
$\alpha|010\rangle + \beta|101\rangle$
$\alpha|100\rangle + \beta|011\rangle$

DIAGNOSE BY MEAS.

$d_1 = x_2 \oplus x_1$	$d_0 = x_1 \oplus x_0$	APPLY CORRECTION
0	0	I
0	1	X_0
1	1	X_1
1	0	X_2

Circuit for 3-qubit bit flip error correction



Summary

- You can not clone (copy) qubits. If you have a large supply of qubits in the same state, the best you can do is to make several measurements in order to determine the state *approximately* (quantum tomography).
- However, you can teleport. If Alice and Bob share an entangled state, Alice can transfer her state into Bob's qubit by making measurements on her state and communicating the outcome via a classical channel (phone call) to Bob. Bob uses this information to transform his qubit in a state identical to Alice's. However, Alice's qubit is now reset to |0> or |1>.
- A related procedure enables quantum error correction. The qubit is encoded into a highly entangled state replacing |0> by |000>, |1> by |111>. After it goes through a noisy channel it can be decoded according to the majority voting rule by measuring correlations between the qubits. The measurement is done with the help of two ancilla qubits – that way the measurement leaves the qubit state uncorrupted! The ability to do quantum error correction suggests that large scale QC can become a practical technology in the future.