# Application to linear algebra

Eigenvalue estimation for unitary operators
(phase estimation)

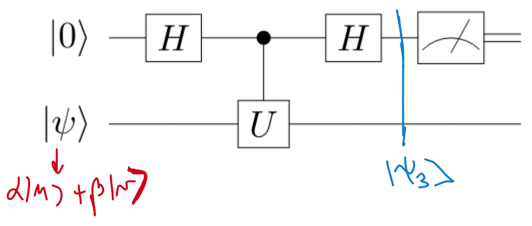
## Circuit that measures the eigenvalue of a Pauli U

- Consider a 1-qubit unitary U with eigenvectors |u> and |v>:

$$U|u\rangle = e^{2\pi\phi_u i}|u\rangle$$
$$U|v\rangle = e^{2\pi\phi_v i}|v\rangle$$

$\phi_u, \phi_v \in [0,1)$

- To start simple, let's take U to be a Pauli matrix. Eigenvalues are +1 and -1 so $\phi_u$=0 and $\phi_v$=1/2. What happens when we input |$\psi$>=α|u> + β|v> to the circuit below?

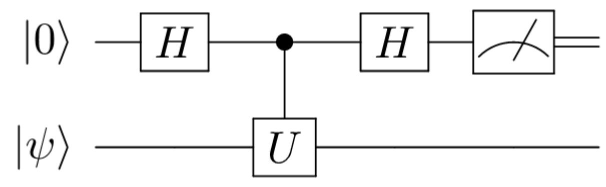



$= |u\rangle$ if $j=0$
$= |v\rangle$ if $j=1$

$$|\psi_3\rangle = \alpha\underbrace{\left(\frac{1+e^{2\pi\phi_u i}}{2}\right)}_{=1}|0\rangle|u\rangle + \beta\underbrace{\left(\frac{1+e^{2\pi\phi_v i}}{2}\right)}_{=0}|0\rangle|v\rangle + \alpha\underbrace{\left(\frac{1-e^{2\pi\phi_u i}}{2}\right)}_{=0}|1\rangle|u\rangle + \beta\underbrace{\left(\frac{1-e^{2\pi\phi_v i}}{2}\right)}_{=1}|1\rangle|v\rangle$$
$$= \alpha|0\rangle|u\rangle + \beta|1\rangle|v\rangle$$

## Implement for U=Y in IBM-Q

(use cy(q_control, q_target))




- Check that you get the right answer if you feed |$\psi$>=|+i> or |-i>.
- Check that you get the right statistics if you feed |$\psi$>=α|+i>+β|-i> for a choice of α,β.

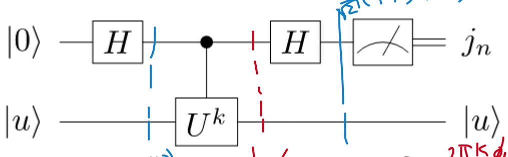
## Kitaev's algorithm for phase estimation

- Write $\phi_u$ as a base-two decimal:

$$\phi_u = 0.j_1j_2j_3\cdots j_n = \frac{j_1}{2} + \frac{j_2}{2^2} + \frac{j_3}{2^3} + \cdots \frac{j_n}{2^n}$$

$\phi_u = 0.0 \hat{=} 0$
$\phi_v = 0.1 \hat{=} \frac{1}{2}$

- Show that this circuit outputs $j_n$ when $k=2^{n-1}$:




$\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)|u\rangle$ $\quad \frac{1}{\sqrt{2}}\left(|0\rangle|u\rangle + |1\rangle e^{2\pi k\phi_u i}|u\rangle\right)$

$$\frac{1}{\sqrt{2}}\left(|+\rangle|u\rangle + e^{2\pi k\phi_u i}|-\rangle|u\rangle\right) = \left\{\left(\frac{1+e^{2\pi k\phi_u i}}{2}\right)|0\rangle|u\rangle + \left(\frac{1-e^{2\pi k\phi_u i}}{2}\right)|1\rangle|u\rangle\right\}$$
$$= e^{\pi k\phi_u i}\left\{\cos(k\pi\phi_u)|0\rangle - i\sin(k\pi\phi_u)|1\rangle\right\}|u\rangle$$
$$= e^{\pi k\phi_u i}\left[R_x(k\,2\pi\phi_u)|0\rangle\right]|u\rangle$$

$k=2^{n-1}$

$$= 2^{n-1}2\pi\left(\frac{j_1}{2} + \frac{j_2}{2^2} + \dots + \frac{j_n}{2^n}\right)$$
$$= 2\pi\left(\underbrace{2^{n-2}j_1 + 2^{n-3}j_2 + \dots}_{\text{integers!}} + \frac{j_n}{2}\right)$$

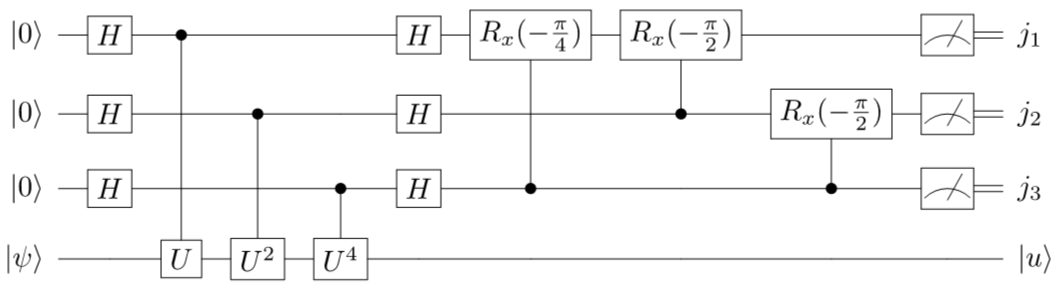
$\Rightarrow R_x(2^{n-1}2\pi\phi_u) = \pm R_x(\pi j_n)$ !

- Test this in IBM-Q for U=$R_y$(θ). Note eigenvalues are

$$\phi_{+i} = 1 - \frac{\theta}{4\pi}$$
$$\phi_{-i} = \frac{\theta}{4\pi}$$

## Three-bit phase estimation

$0.110 \leq 0.1101 < 0.111$




- The circuit above outputs $\phi_u$=0.$j_1j_2j_3$ for one of the |u> contained in |$\psi$> (|u> is sampled randomly out of all |u>'s overlapping with |$\psi$>).
- Implement the circuit above for U=$R_y$(θ) and check that it works for particular choices of θ/(4π) = 0.$j_1j_2j_3$ .
- Try θ/(4π) = 0.$j_1j_2j_3$ – ε, with ε small. If you run the circuit many times does it return the 0.$j_1j_2j_3$ closest to the actual answer θ/(4π) ?

## Let's get serious?

- Increase the size of U to a 4X4 matrix by using two qubits in the input register.
- Use Toffoli gate to implement control U.