Adiabatic quantum computing

Alternative to the standard "gate model" of quantum computing

The Hamiltonian operator: Represents energy in quantum theory

- We learned that Hermitian operators describe "observables": things that we can measure. ENERGY is described by an operator called "Hamiltonian", denoted $\mathcal{H}$ (H in calligraphic font – not to be confused with Hadamard!).
- Example: Single qubit, like an "artificial atom".

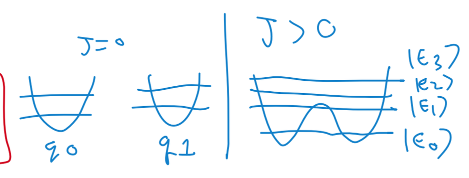
$$\mathcal{H}_1 = \frac{\hbar\omega}{2}(I - Z) = \hbar\omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$E_1 = \hbar\omega$, $|1\rangle$; $E_0 = 0$, $|0\rangle$; $\hbar\omega$

$\hbar = 6.62\times10^{-34}$ Js, ω is frequency, $\omega = 5-7$ GHz for IBM-Q.

- Two qubits: Artificial molecule. (e.g. H_2)

$$\mathcal{H}_2 = \frac{\hbar\omega}{2}\sum_{i=0}^{1}(I - Z_i) + \frac{\hbar J}{4}(X_0X_1 + Y_0Y_1 + Z_0Z_1)$$



Eigenstates of the Hamiltonian: Ground and excited states

- The eigenstates (= eigenvector) of $\mathcal{H}$ have "well defined energies" set by the associated eigenvalue. The eigenstate associated to the lowest eigenvalue is called *ground state*, denoted $|E_0\rangle$. These states are special because they are quite stable: They can only change if energy is injected in them.

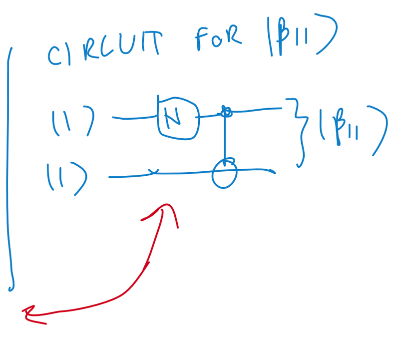
$$\mathcal{H}_1 = \frac{\hbar\omega}{2}(I - Z) = \hbar\omega \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$|E_1\rangle = |1\rangle$, $E_1 = \hbar\omega$; $|E_0\rangle = |0\rangle$, $E_0 = 0$ for $\omega < J$

$$\mathcal{H}_2 = \begin{pmatrix} J/4 & 0 & 0 & 0 \\ 0 & (\omega - J/4) & J/2 & 0 \\ 0 & J/2 & (\omega - J/4) & 0 \\ 0 & 0 & 0 & (2\omega + J/4) \end{pmatrix}$$

$|11\rangle = |E_3\rangle$; $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |E_2\rangle$; $|00\rangle = |E_1\rangle$; $|E_0\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\beta_{11}\rangle$

Circuit for $|\beta_{11}\rangle$



- The other states with higher energy are called *excited states*.

Time evolution of a quantum state

- The time evolution of a quantum state is dictated by Schröedinger's eqn, (the equivalent of Newton's F=ma for quantum mechanics!):

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle$$

→ a system of n 1st order differential eqns.

- When $\mathcal{H}$ is independent of time, this eqn is quite easy to solve: The general solution is

$$|\psi(t)\rangle = e^{-i\frac{t}{\hbar}\mathcal{H}}|\psi(0)\rangle$$

Time evolution operator

Proof: $i\hbar\frac{\partial}{\partial t}\left[e^{-\frac{i}{\hbar}t\mathcal{H}}|\psi(0)\rangle\right] = i\hbar\left[e^{-\frac{i}{\hbar}t\mathcal{H}}\frac{\partial}{\partial t}\left(-\frac{i}{\hbar}t\mathcal{H}\right)\right]|\psi(0)\rangle = i\hbar\left(-\frac{i}{\hbar}\mathcal{H}\right)e^{-\frac{i}{\hbar}t\mathcal{H}}|\psi(0)\rangle = \mathcal{H}\left[e^{-\frac{i}{\hbar}t\mathcal{H}}|\psi(0)\rangle\right] = \mathcal{H}|\psi(t)\rangle$

Time evolution leads to interference and *tunnelling*

- If $|\psi(0)\rangle$ is an eigenvector of $\mathcal{H}$, $|\psi(t)\rangle$ only gets a global phase:

$\mathcal{H}|\psi(0)\rangle = E_j|\psi(0)\rangle \Rightarrow |\psi(t)\rangle = e^{-\frac{i}{\hbar}t\mathcal{H}}|\psi(0)\rangle = e^{-\frac{i t E_j}{\hbar}}|\psi(0)\rangle$; $|\psi(0)\rangle = |E_j\rangle$

Apart from global phase, $|\psi(t)\rangle$ is the same as $|\psi(0)\rangle$!

- Consider what happens when $|\psi(0)\rangle = \alpha|E_1\rangle + \beta|E_2\rangle$:

$|\psi(t)\rangle = e^{-\frac{i}{\hbar}t\mathcal{H}}|\psi(0)\rangle = \alpha e^{-\frac{itE_1}{\hbar}}|E_1\rangle + \beta e^{-\frac{itE_2}{\hbar}}|E_2\rangle = e^{-\frac{itE_1}{\hbar}}\left[\alpha|E_1\rangle + \beta e^{-\frac{i(E_2-E_1)t}{\hbar}}|E_2\rangle\right]$

Phase difference, interference

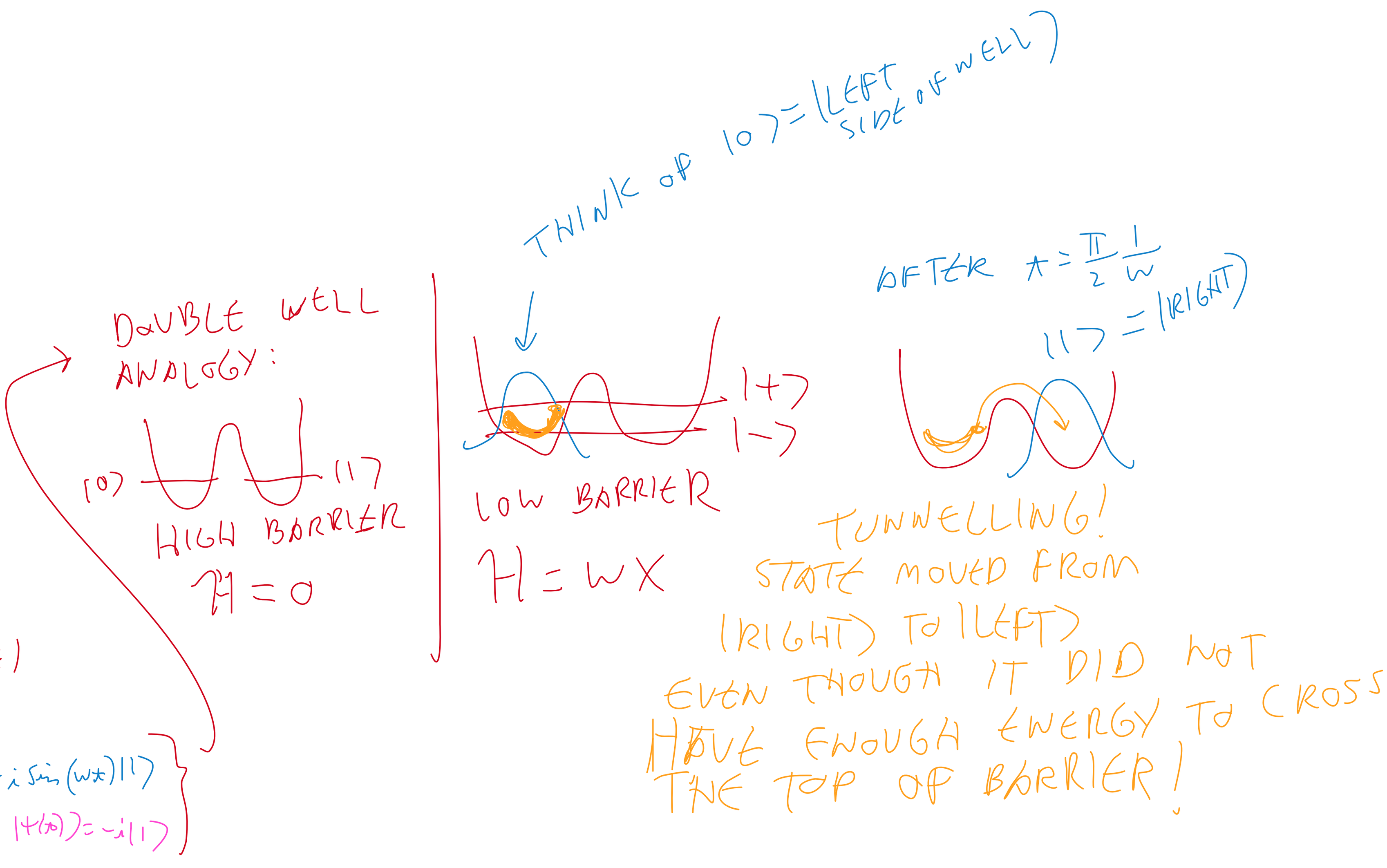
- Example: $\mathcal{H} = \omega X$ (from now on, omit $\hbar$ and measure energy in frequency units!)

E_j: $+\omega$, $|+\rangle$; $-\omega$, $|-\rangle$

Consider $|\psi(0)\rangle = |0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$

$|\psi(t)\rangle = e^{-i t\mathcal{H}}|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\omega t}|+\rangle + e^{i\omega t}|-\rangle\right] = \cos(\omega t)|0\rangle - i\sin(\omega t)|1\rangle$

$\Rightarrow$ at $t = \frac{\pi}{2\omega} \Rightarrow |\psi(t)\rangle = -i|1\rangle$



$\mathcal{H}(t)$ with slow time dependence

- But what if $\mathcal{H}$ changes in time? Sol. of Schröedinger's eqn is much more difficult. But at each time t we have "instantaneous eigenstates" $|E_j(t)\rangle$ satisfying

$$\mathcal{H}(t)|E_j(t)\rangle = E_j(t)|E_j(t)\rangle$$

- Assume $\mathcal{H}(t)$ changes very slowly (adiabatic) and the energy levels E_j do not coincide with each other. In this case, if at t=0 the qubits are in one of the eigenstates $|E_j(0)\rangle$ of $\mathcal{H}(t=0)$, they will remain in the "instantaneous eigenstate" $|E_j(t)\rangle$. Example:

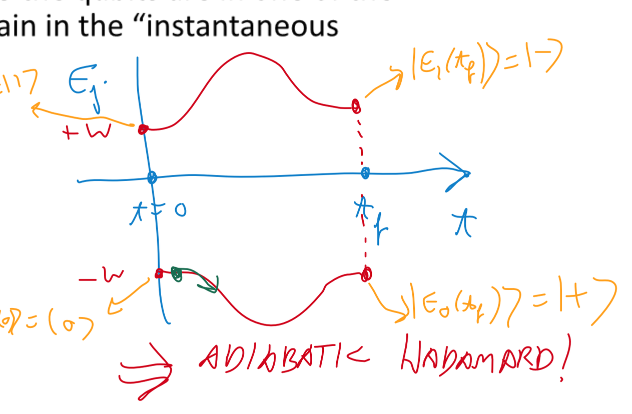
$$\mathcal{H}(t) = -\omega\{[1 - s(t)]Z + s(t)X\}$$

$s(t)$ is called "schedule" function.

$s(0) = 0$; $s(t_f) = 1$

At $t = 0$: $\mathcal{H}(0) = -\omega Z$; $t = t_f$: $\mathcal{H}(t_f) = -\omega X$

E.g. $s(t) = t/t_f$.



Adiabatic theorem

- Let $|E_j(s(t))\rangle$ be the instantaneous eigenstate of $\mathcal{H}(s(t))$ with $E_j(s(t)) \leq E_{j+1}(s(t))$. Assume that at $t = 0$, $s(0) = 0$ and $|\psi(0)\rangle = |E_j(0)\rangle$. Then, if $s(t_f) = 1$ and

$$t_f \gg \text{Max}_{s\in[0,1]}\frac{\hbar\,|\langle E_i(s)|\,\partial\mathcal{H}(s)/\partial s\,|E_j(s)\rangle|}{|E_i(s) - E_j(s)|^2} \quad \text{for } i \neq j,$$

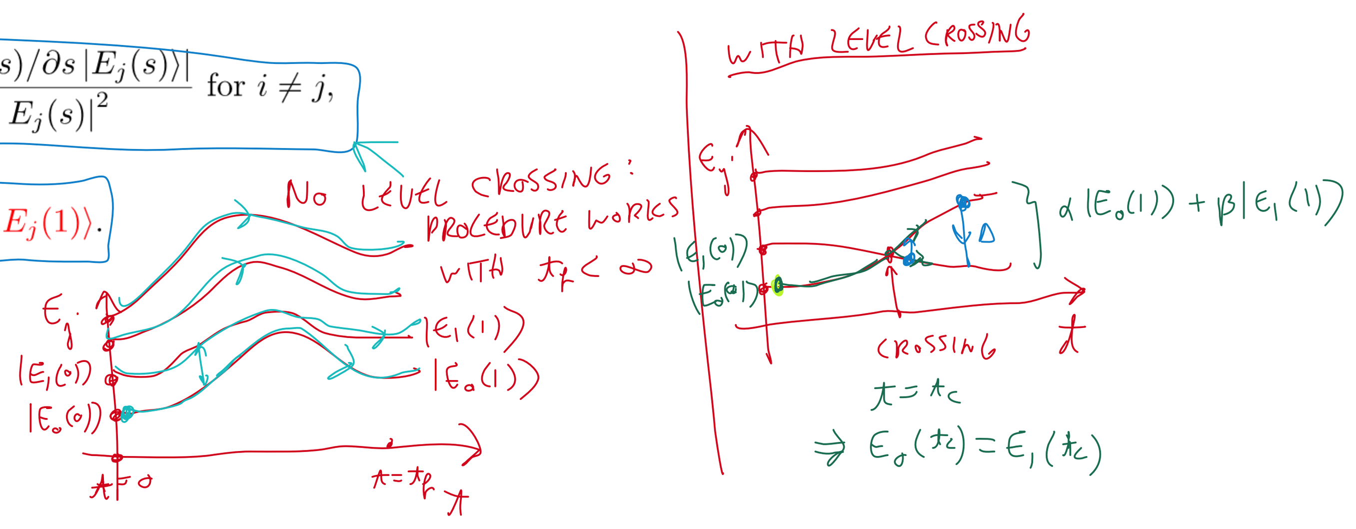
we will have

$$|\psi(t_f)\rangle = e^{i\phi}|E_j(1)\rangle.$$

Assume n qubits interacting with each other:

$\mathcal{H} = \sum_j \omega_j(t) Z_j + \sum_{i,j} a_{ij}(t) X_i X_j$

No level crossing: procedure works with $t_f < \infty$



Adiabatic quantum computation (AQC)

- Engineer a Hamiltonian that changes in time so that $\mathcal{H}(t=0)$ has a known product state as its ground state, e.g.

$$|E_0(0)\rangle = |+\rangle|+\rangle\cdots|+\rangle = \sum_{x\in\{0,1\}^n}|x\rangle$$

- And for t=t_f, $\mathcal{H}(t_f)$ has a ground state that encodes the solution to the problem you want to solve. E.g. for adiabatic Deutch-Josza:

$$|E_0(t_f)\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle$$

- So that performing a measurement (in a particular basis) will give the desired answer to the problem.

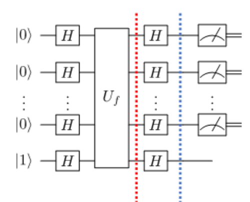
Measure in Hadamard basis $\{|+\rangle, |-\rangle\}$ and get

$P(|00\cdots0\rangle) = 1$ if $f(x) =$ const

$P(|00\cdots0\rangle) = 0$ if $f(x) =$ balanced.

Example: Adiabatic version of Deutch-Josza algorithm

- Recall gate model Deutch-Josza:



$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$|\psi_3\rangle = \sum_{z\in\{0,1\}^n}\left(\frac{1}{2^n}\sum_{x\in\{0,1\}}(-1)^{f(x)+z\cdot x}\right)|z\rangle|1\rangle$$

- Adiabatic version: