1. **Deformation potential in a triply degenerate (p-like) band.**

The application of stress to a crystal can split a set of degenerate bands. Crystal symmetry dictates how each band is coupled to the other bands, i.e., how the deformation interaction depends on the elements of the strain tensor. In a cubic crystal, a triply degenerate band with p-symmetry has the following deformation potential Hamiltonian, known as the Pikus-Bir Hamiltonian:

\[ H_{PB} = a \text{Tr}(\epsilon) \mathbf{1}_{3 \times 3} + \begin{pmatrix}
    b(\epsilon_{xx} + \epsilon_{yy} - 2\epsilon_{zz}) & \sqrt{\frac{3}{2}}d(\epsilon_{xz} + i\epsilon_{yz}) & \sqrt{\frac{3}{2}}d(-\epsilon_{xz} + i\epsilon_{yz}) \\
    \sqrt{\frac{3}{2}}d(\epsilon_{xz} - i\epsilon_{yz}) & -\frac{1}{2}b(\epsilon_{xx} + \epsilon_{yy} - 2\epsilon_{zz}) & 3b(\epsilon_{xx} - \epsilon_{yy}) - i\sqrt{3}d\epsilon_{xy} \\
    \sqrt{\frac{3}{2}}d(-\epsilon_{xx} - i\epsilon_{yz}) & \frac{3}{2}b(\epsilon_{xx} - \epsilon_{yy}) + i\sqrt{3}d\epsilon_{xy} & -\frac{1}{2}b(\epsilon_{xx} + \epsilon_{yy} - 2\epsilon_{zz})
\end{pmatrix}, \]

operating on the basis of the three degenerate “p-like” \((L = 1)\) and \(k = 0\) band states \(|0⟩ = |z⟩\), \(|1⟩ = \frac{1}{\sqrt{2}}(|x⟩ + i|y⟩)\), and \(|−1⟩ = \frac{1}{\sqrt{2}}(|x⟩ − i|y⟩)\). The constant \(a\) is a hydrostatic deformation potential, while \(b\) and \(d\) are shear deformation potentials. Note that this becomes an electron-phonon interaction when we plug the strain as a function of the phonon creation and anihilation operators.

(a) Assume uniaxial strain along one of the [001] directions, i.e. \(\epsilon_{\alpha\alpha} = \epsilon\) for only one \(\alpha\), all other strains equal to zero. Find the eigenenergies of the Hamiltonian, and show that the hydrostatic energy shift is \(a\epsilon\), while the energy splitting of the states is equal to \(|3b\epsilon|\).

(b) For uniaxial strain along the [111] direction given by \(\epsilon'\), show that the splitting of the states is equal to \(|\sqrt{3}d\epsilon'|\).

(c) Show that this Hamiltonian gives a nonzero electron-phonon interaction for longitudinal phonons – phonons with polarization along \(k\). Show that the shear term also gives a non-zero contribution for transverse phonons.

_Hint: Uniaxial strain along the [111] direction is given by_

\[ \epsilon = \epsilon' \begin{pmatrix} \hat{x} + \hat{y} + \hat{z} \\ \sqrt{3} \end{pmatrix} \begin{pmatrix} \hat{x} + \hat{y} + \hat{z} \\ \sqrt{3} \end{pmatrix}. \]
2. Temperature dependence of the electron-phonon scattering time in a metal at low temperature.

The electron-phonon scattering process has to obey the following conservation rules:

\[ q = q' \pm k \]  \hspace{1cm} \text{(2)}

\[ E_q = E_{q'} \pm \hbar \omega_k. \]  \hspace{1cm} \text{(3)}

Moreover, at low temperatures (much smaller than the Debye temperature), the physically relevant phonons for e-ph scattering have wavevectors \( k \) of the order of \( k_B T / (\hbar \nu) \). This is because the phonons that are available for absorption are the thermally excited ones; and the phonons that are generated by spontaneous emission can not have energy much larger than \( k_B T \) (due to Pauli exclusion electrons can only give away energy of the order of \( k_B T \), the smearing of the Fermi sea).

(a) Use this information to draw a diagram of electron-phonon scattering in a spherical Fermi surface, i.e., for a given electron \( q \), draw the region were \( q' \) can be located. Conclude that the sum of these \( q' \)'s, “the phase space for scattering” has to scale as \( T^2 \).

(b) How does the rate for e-ph scattering due to the deformation potential interaction scales with temperature?

(c) How does the rate for e-ph scattering due to the piezo-electric interaction scales with temperature?

3. Length scale for the absorption of light in a semiconductor.

The rate for interband transitions induced by photons in a semiconductor was derived to be

\[ \frac{1}{\tau} = \sqrt{\hbar \omega - E_{\text{gap}}} \frac{\epsilon^2 m_r^{3/2}}{\pi \hbar^2 \epsilon m_e^2} |\langle v | p | c \rangle \cdot \hat{n} |^2 \hbar \omega}. \]  \hspace{1cm} \text{(4)}

(a) Convert this to an absorption length. That is, show that for a constant fluence of photons hitting a surface at \( x = 0 \), the probability of a photon not being absorbed at distance \( x \) from the surface is given by

\[ P(x) = e^{-x/l}, \]  \hspace{1cm} \text{(5)}

where \( l \) is the absorption length. Write down an expression for \( l \) using Eq. (4).

\text{Hint: Once inside the material, the photon moves with velocity } c/n. \text{ Assume the photon frequency is given by } \hbar \omega / E_{\text{gap}} = 1.1. \text{ The matrix element for a semiconductor is}

\[ m_r \text{ is the effective mass, } m_e \text{ is the free electron mass.} \]
usually given in terms of the unitless oscillator strength:

\[ f = \frac{2|\langle v|p|c\rangle|^2}{mE_{\text{gap}}}, \quad (6) \]

which is of the order of unity for typical semiconductors.

4. **Gr"uneisen parameters in one dimension.**

Consider a one-dimensional array of \( N \) atoms interacting through pair potentials \( \phi(r) \). In this problem you will include interaction between first and second nearest neighbor atoms.

**(a)** Compute the frequency of normal modes of vibration \( \omega_k \) as a function of \( \phi''(a) \) and \( \phi''(2a) \), where \( a \) is the equilibrium interatomic distance.

**(b)** Assume that the array is subject to hydrostatic strain \( \varepsilon \), i.e., its equilibrium interatomic distance changes to \( (1 + \varepsilon)a \). Compute the \( k \)-dependent Gr"uneisen parameter, defined by

\[ \gamma_k = \frac{1}{\omega_k} \frac{\partial \omega_k}{\partial \varepsilon} \bigg|_{\varepsilon=0}. \quad (7) \]

Show that \( \gamma_k \) is actually independent of \( k \) when only nearest-neighbor interactions are present.

*Hint: Don’t make the long wavelength approximation here (i.e., don’t drop terms of order \( k^2 \)). Label your normal modes by \( k(1+\varepsilon)a \rightarrow k \), i.e., the phase difference from one atom to the other. Equation (7) is the volume dependence of the same mode (same interatomic phase difference). If \( k \) depends on \( \varepsilon \), we will get an additional (incorrect) contribution to \( \gamma_k \).*