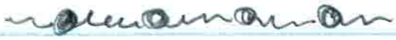
Lecture # 16. Phonons and photons



Review:

$$\mathcal{H} = \sum_n \left[\frac{p_n^2}{2m} + \frac{1}{2} K (x_n - x_{n-1})^2 \right]$$

Found normal modes (in this case, simple Fourier transform)

$$\Rightarrow \begin{cases} x_k = \dfrac{1}{\sqrt{N}} \displaystyle\sum_n x_n e^{-ikan} & (x_k^\dagger = x_{-k}) \\ p_k = \dfrac{1}{\sqrt{N}} \displaystyle\sum_n p_n e^{-ikan} & (p_k^\dagger = p_{-k}) \end{cases}$$

$$\Rightarrow \mathcal{H} = \sum_k \left(\frac{|p_k|^2}{2m} + \frac{1}{2} m \omega_k^2 |x_k|^2 \right)$$

Define $\hat{a}_k = \dfrac{1}{\sqrt{2}} \left[\underbrace{\left(\frac{1}{\ell}\right) x_k}_{\hat{x}_k} + i \underbrace{\frac{1}{\left(\frac{\ell}{\hbar}\right)} p_k}_{\hat{p}_k} \right]$ $\left(\ell = \sqrt{\dfrac{\hbar}{m\omega_k}} \right)$

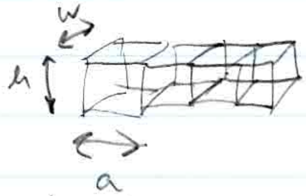
$$\Rightarrow [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{k,k'} \quad \text{and}$$

$$\boxed{\mathcal{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right)}$$

$$\mathcal{H} \underbrace{| \ldots, n_k, \ldots \rangle}_{\substack{\text{Fock states,} \\ \text{or "number states".}}} = \sum_k \hbar\omega_k \left(n_k + \frac{1}{2} \right) | \ldots, n_k, \ldots \rangle$$

Eigenstates of $a^\dagger a$

Continuum limit:



$$m = \rho V_{cell} = \rho(a\underbrace{hw}_{A})$$

$$\sigma = \frac{F}{A} = C\varepsilon = -C\frac{x}{a} \leftrightarrow F = -Kx \Rightarrow K = C\frac{A}{a} = C\frac{hw}{a}$$

$$\omega_k = 2\sqrt{\frac{K}{m}}\left|\sin\left(\frac{ka}{2}\right)\right| = 2\sqrt{\frac{C\cancel{hw}}{a}\frac{1}{\rho a \cancel{hw}}}\left|\sin\left(\frac{ka}{2}\right)\right| = 2\sqrt{\frac{C}{\rho}}\frac{\sin\left(\frac{ka}{2}\right)}{a}$$

Take $a \to 0$: $\omega_k \to 2\sqrt{\frac{C}{\rho}}\frac{k\cancel{a}}{2}\frac{1}{\cancel{a}} = \underbrace{\sqrt{\frac{C}{\rho}}}_{v}k$

From $x_n = \frac{1}{\sqrt{N}}\sum_k x_k e^{ikan}$ and

$$\begin{cases} \hat{a}_k = \frac{1}{\sqrt{2}}\left(\frac{1}{\ell}x_k + i\frac{1}{\hbar/\ell}p_k\right) \\ \hat{a}^\dagger_{-k} = \frac{1}{\sqrt{2}}\left(\frac{1}{\ell}x^\dagger_{-k} - i\frac{1}{\hbar/\ell}p^\dagger_{-k}\right) = \frac{1}{\sqrt{2}}\left(\frac{x_{+k}}{\ell} - i\frac{\ell}{\hbar}p_{+k}\right) \end{cases}$$

$$\Rightarrow x_k = \frac{\ell}{\sqrt{2}}\left(\hat{a}^\dagger_k + a^\dagger_{-k}\right)$$

$$\Rightarrow x_n = \frac{\ell}{\sqrt{2N}}\sum_k\left(a^\dagger_k + a^\dagger_{-k}\right)e^{ikan} = \frac{\ell}{\sqrt{2N}}\sum_k\left(\hat{a}^\dagger_k e^{ikan} + \hat{a}^\dagger_k e^{-ikan}\right)$$

Take $a \to 0$ (continuum): $na \to r$

$$x(r) = \underbrace{\sqrt{\frac{\hbar}{2Nm\omega_k}}}\sum_k\left(a^\dagger_k e^{ikr} + \hat{a}^\dagger_k e^{-ikr}\right)$$

$$2Nm\omega_k = 2N(\rho V_{cell})\omega_k = 2\underbrace{V}_{NV_{cell}}\rho\omega_k$$

$$\boxed{x(r) = \sqrt{\frac{\hbar}{2V\rho\omega_k}}\sum_k\left(a^\dagger_k e^{ikr} + a^\dagger_k e^{-ikr}\right)}$$

"Quantum string"

Also, we can do the same for momentum :

$$\frac{p(r)}{m} = \frac{1}{m}\frac{1}{\sqrt{N}} \sum_k \left(\frac{-i}{\sqrt{2}}\frac{\hbar}{l}\right)\left(a_k - a^{+}_{-k}\right) e^{ikr}$$

$$\frac{p(r)}{m} = \underbrace{\frac{1}{m}\frac{1}{\sqrt{2N}}\sqrt{\hbar m w_k}}_{\sqrt{\frac{\hbar w_k}{2Nm}} = \sqrt{\frac{\hbar w_k}{2\rho V}}} (-i)\sum_k \left(\hat{a}_k e^{ikr} - \hat{a}^{+}_k e^{-ikr}\right)$$

$$\boxed{\dot{x}(r) = \frac{p(r)}{m} = -i\sum_k \sqrt{\frac{\hbar w_k}{2\rho V}}\left(\hat{a}_k e^{ikr} - \hat{a}^{+}_k e^{-ikr}\right)}$$

in 3 dimensions:

$$\vec{x}(\vec{r}) = \sum_{\vec{k},\lambda} \hat{n}_{\vec{k},\lambda} \sqrt{\frac{\hbar}{2V\rho w_{k\lambda}}}\left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} + \hat{a}^{+}_{k\lambda} e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\dot{\vec{x}}(r) = -i \sum_{\vec{k},\lambda} \hat{n}_{\vec{k},\lambda} \sqrt{\frac{\hbar w_{k\lambda}}{2\rho V}}\left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} - \hat{a}^{+}_{k\lambda} e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\lambda = L, T1, T2 .$$

Nov. 1st

Photons

Maxwell's eqns in S.I.:

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \end{cases}$$

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$ (Because $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ for all $\vec{A}$)

$\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A} \Rightarrow \vec{\nabla}\times\left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right) = \vec{0} \Rightarrow \left(\vec{E} + \frac{\partial\vec{A}}{\partial t}\right)$ is "Conservative".

$\Rightarrow \vec{E} + \frac{\partial\vec{A}}{\partial t} = -\vec{\nabla}\phi$

Chose Coulomb Gauge: $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{\nabla}\cdot\vec{E} = -\nabla^2\phi = \frac{\rho}{\varepsilon_0} \Rightarrow \phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}\,')d^3r'}{|\vec{r}-\vec{r}\,'|}$

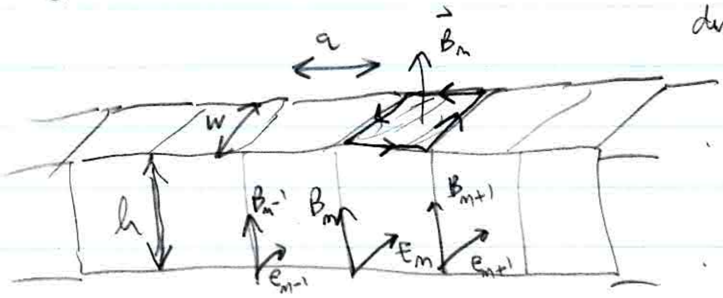
In vaccum, $\rho = 0$ and $\vec{J} = 0$:

$$\phi = 0, \quad \vec{E} = -\frac{\partial\vec{A}}{\partial t}, \text{ and } \vec{B} = \vec{\nabla}\times\vec{A}$$

Energy density of EM field:

$$H = \int d^3r\left[\frac{1}{2}\varepsilon_0|\vec{E}|^2 + \frac{1}{2}\frac{1}{\mu_0}|\vec{B}|^2\right] = V_{cell}\sum_n\left[\frac{1}{2}\varepsilon_0 E_n^2 + \frac{1}{2}\frac{1}{\mu_0}B_n^2\right]$$

discretization of vaccum



From $\vec{B} = \nabla \times \vec{A} \Rightarrow \int \vec{B} \cdot d\vec{\sigma} = \oint d\vec{l} \cdot \vec{A}$

Apply this to cell indicated in Figure:

$$B_n w a = (A_{n+1} - A_n) w \Rightarrow B_n = \left(\frac{A_{n+1} - A_n}{a} \right)$$

Also, $\vec{E} = -\frac{\partial \vec{A}}{\partial t} \Rightarrow E_n = -\dot{A}_n$

$$\Rightarrow \mathcal{H} = V_{cell} \sum_n \left[\frac{1}{2} \varepsilon_0 (\dot{A}_n)^2 + \frac{1}{2} \frac{1}{\mu_0 a^2} (A_{n+1} - A_n)^2 \right]$$

The point is that $\mathcal{H}$, the energy density of EM field actually does look like a set harmonic oscillators when we write it in terms of $\vec{A}$ only!

Just like sound wave!

Compare: $\mathcal{H} = \sum_n \left[\frac{1}{2} m (\dot{x}_n)^2 + \frac{1}{2} K (x_{n+1} - x_n)^2 \right]$.

Substitute
$$\begin{cases} x_n \leftrightarrow A_n, \quad \dot{x}_n \longleftrightarrow E_n \\ m \leftrightarrow V_{cell} \varepsilon_0 \\ K \leftrightarrow \frac{V_{cell}}{\mu_0 a^2} \end{cases}$$

And get

$$\Rightarrow \mathcal{H} = \sum_k \left(a_k^\dagger a_k + \tfrac{1}{2} \right) \hbar\omega$$

With $\omega = 2\sqrt{\frac{K}{m}} \left| \sin\left(\frac{ka}{2}\right) \right| = 2 \sqrt{\frac{V_{cell}}{\mu_0 a^2 V_{cell} \varepsilon_0}} \left| \sin\left(\frac{ka}{2}\right) \right| \overset{a \to 0}{=} \frac{2c}{a} \frac{ka}{2} = ck$ //

$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$$a_k = \frac{1}{\sqrt{2}} \left(\sqrt{\frac{\varepsilon_0 V_{cell} \omega}{\hbar}} A_k + i \sqrt{\frac{\varepsilon_0 V_{cell}}{\hbar \omega}} E_k \right)$$

$$\vec{A}(\vec{r}) = \sum_{\vec{k},\lambda} \hat{n}_{\vec{k},\lambda} \sqrt{\frac{\hbar}{2\varepsilon_0 V \omega_k}} \left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} + \hat{a}^{\dagger}_{k\lambda} e^{-i\vec{k}\cdot\vec{r}} \right)$$

$$\vec{E}(\vec{r}) = -i \sum_{\vec{k},\lambda} \hat{n}_{k\lambda} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} - a^{\dagger}_{\vec{k}\lambda} e^{-i\vec{k}\cdot\vec{r}} \right)$$

$\lambda = 1, 2$ only transverse modes.