Lecture #17: Coherent states, field operators

What do operators $\hat{a}_k$ measure? They actually correspond to the complex amplitude ($\alpha_k = A_k e^{i\theta_k}$) of a wave. Consider the eigenstate of $\hat{a}_k$:

$$\hat{a}_k \underbrace{|\alpha_k\rangle}_{\text{"Coherent state"}} = \underbrace{\alpha_k}_{A_k e^{i\theta_k}} |\alpha_k\rangle \quad \text{and} \quad \langle\alpha_k| \hat{a}_k^\dagger = \langle\alpha_k| \alpha_k^*$$

$\hat{a}_k$ is non-hermitian, it can't be measured. However, $x_k \sim \hat{a}_k + \hat{a}_k^\dagger$ can:

$$\langle\alpha_k|(\hat{a}_k + \hat{a}_k^\dagger)|\alpha_k\rangle = \alpha_k + \alpha_k^* = 2A_k \cos(\theta_k)$$

Also,

$$\langle\hat{N}_k\rangle = \langle\alpha_k| \hat{a}_k^\dagger \hat{a}_k |\alpha_k\rangle = |\alpha_k|^2 = A_k^2$$, i.e. amplitude of wave measures the number of photons (phonons) in the coherent state.

How does $|\alpha_k\rangle$ relates to $|n_k\rangle$?

In the homework you will show that

$$|\alpha_k\rangle = e^{-\frac{|\alpha_k|^2}{2}} \sum_{n_k=0}^{\infty} \frac{(\alpha_k)^{n_k}}{\sqrt{n_k!}} |n_k\rangle$$

Considering that $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle \Rightarrow |\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi(0)\rangle$

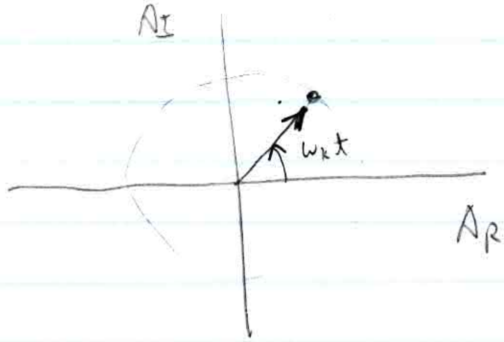
$$H = \hbar\omega_k (a_k^\dagger a_k + \tfrac{1}{2}) \Rightarrow |\alpha_k(t)\rangle = e^{-\frac{|\alpha_k|^2}{2}} \sum_{n_k=0}^{\infty} \frac{(\alpha_k e^{-i\omega_k t})^{n_k}}{\sqrt{n_k!}} |n_k\rangle$$

$$= |\alpha_k e^{-i\omega_k t}\rangle$$

Let's represent a coherent state by its phase space diagram:

$$\langle \alpha_k | x_k | \alpha_k \rangle \propto \langle \alpha_k | \left(\frac{a_k + a_k^\dagger}{2} \right) | \alpha_k \rangle = A_k \cos(\omega_k t + \theta_k) = A_R$$

$$\langle \alpha_k | p_k | \alpha_k \rangle \propto \langle \alpha_k | \left(\frac{a_k - a_k^\dagger}{2i} \right) | \alpha_k \rangle = A_k \sin(\omega_k t + \theta_k) = A_I$$



"Phasor picture"

The uncertainty in x is given by:

$$\langle \alpha_k | (\Delta \hat{A}_R)^2 | \alpha_k \rangle = \langle \alpha_k | (\hat{A}_k - \bar{A}_R)^2 | \alpha_k \rangle$$

$$= \langle \alpha_k | \hat{A}_R^2 | \alpha_k \rangle - \bar{A}_R^2$$

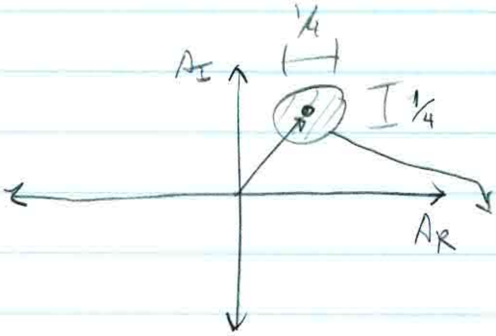
$$= \langle \alpha_k | \left(\frac{a_k^2 + \underbrace{a_k a_k^\dagger + a_k^\dagger a_k}_{1+2a^\dagger a} + (a_k^\dagger)^2}{4} \right) | \alpha_k \rangle - A_k^2 \cos^2(\omega_k t)$$

$$= \frac{1}{4} \left\{ A_k^2 e^{2i\omega_k t} + 1 + 2A_k^2 + A_k^2 e^{-2i\omega_k t} \right\} - A_k^2 \cos^2(\omega_k t)$$

$$= \frac{1}{4} + \frac{A_k^2}{4} \left[2 + \underbrace{2\cos(2\omega_k t)}_{2\cos^2(\omega_k t) - 2\sin^2(\omega_k t)} - 4\cos^2(\omega_k t) \right]$$

$$= \frac{1}{4} + \frac{A_k^2}{4} \underbrace{\left[2 - 2\cos^2(\omega_k t) - 2\sin^2(\omega_k t) \right]}_{=0} = \frac{1}{4}$$

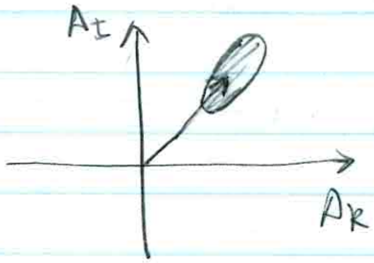
Similarly, $\langle (\Delta \hat{A}_I)^2 \rangle = \frac{1}{4}$.



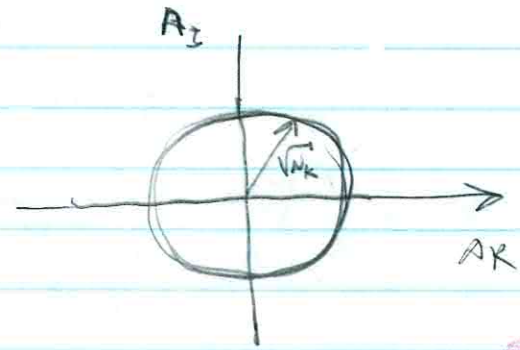
Uncertainty principle:

$$(\Delta A_R)(\Delta A_I) = \frac{1}{4}$$

Area remains fixed! We can squeeze the state by decreasing uncertainty in one amplitude at the expense of the other. These are called squeezed states.



"Squeezed" state
One axis of uncertainty traded off against each other.



Fock state: $|N_K\rangle$ has well defined amplitude $A_K = \sqrt{N_K}$ but "randomized" phase.
(The probability of finding phase θ_K is uniform in $[0, 2\pi)$!)

A coherent state represents the classical limit: For phonons:

$$\langle \hat{x}(r,t) \rangle = \sum_{k'} \sqrt{\frac{\hbar}{2\rho V \omega_{k'}}} \langle \alpha_k(t) | \left(a_{k'} e^{ikr} + a^\dagger_{k'} e^{-ikr} \right) | \alpha_k(t) \rangle$$

$$= \sum_{k'} \sqrt{\frac{\hbar}{2\rho V \omega_{k'}}} \left[A_k e^{i(kr - \omega_k t + \theta_k)} + A_k e^{-i(kr - \omega_k t + \theta_k)} \right] \delta_{k,k'}$$

$$\langle \hat{x}(r,t)\rangle = \sqrt{\frac{\hbar}{2\rho V \omega_k}}\; 2A_k \cos(kr - \omega_k t + \theta_k)$$

$$= \underbrace{x_0}_{\text{"amplitude of classical wave"}} \cos(kr - \omega_k t + \theta_k)$$

$$x_0 = \sqrt{\frac{2\hbar}{\rho V \omega_k}}\; A_k \;\Rightarrow\; \boxed{A_k = \sqrt{\frac{\rho V \omega_k}{2\hbar}}\; x_0} \quad \text{or} \quad \boxed{\langle \hat{N}_k \rangle = A_k^2 = \left(\frac{\rho V \omega_k}{2\hbar}\right) x_0^2}$$

Connection between number of phonon quanta and classical amplitudes.

Similarly, for photons,

Photons

$$\boxed{A_k = \sqrt{\frac{\varepsilon_0 V}{2\hbar\omega}}\; E_0} \quad \boxed{\langle \hat{N}_k \rangle = \frac{\varepsilon_0 V}{2\hbar\omega}\, E_0^2}$$

$\omega = ck$

Field theory

Consider a complete Basis set $\{|\phi_k\rangle\}$. Any wave function can be written in terms of it: $|\Psi\rangle = \sum_k |\phi_k\rangle\langle\phi_k|\Psi\rangle$ or $\sum_k |\phi_k\rangle\langle\phi_k| = \mathbb{1}$.

"Completeness".

Also, $|\phi_k\rangle = a_k^\dagger|0\rangle$; this is certainly true for an harmonic oscillator, but can be generalized to all other problems!

Define field operator:

$$\begin{cases} \hat{\Psi}(\vec{r}) = \sum_k \underbrace{\phi_k(\vec{r})}_{\langle\vec{r}|\phi_k\rangle} a_k \\ \hat{\Psi}^\dagger(\vec{r}) = \sum_k \phi_k^*(\vec{r}) a_k^\dagger \end{cases}$$

$$\Psi^\dagger(\vec{r})|0\rangle = \sum_k \langle\phi_k|\vec{r}\rangle \underbrace{a_k^\dagger|0\rangle}_{|\phi_k\rangle} = \sum_k \langle\phi_k|\vec{r}\rangle|\phi_k\rangle = \underbrace{\sum_k |\phi_k\rangle\langle\phi_k|}_{1}\vec{r}\rangle$$

$$= |\vec{r}\rangle$$

$\Psi^\dagger(r)$ creates a particle localized at point r!

$$[\Psi(\vec{r}), \Psi^\dagger(\vec{r}')] = \sum_{k,k'} \phi_k(\vec{r})\phi_{k'}^*(\vec{r}')\underbrace{[a_k, a_{k'}^\dagger]}_{\delta_{k,k'}} = \sum_k \phi_k(\vec{r})\phi_k^*(\vec{r}')$$

$$= \sum_k \langle\vec{r}|\phi_k\rangle\langle\phi_k|\vec{r}'\rangle = \langle\vec{r}|\vec{r}'\rangle = \delta(\vec{r}-\vec{r}')$$

$$\Longrightarrow \boxed{[\Psi(\vec{r}), \Psi^\dagger(\vec{r}')] = \delta(\vec{r}-\vec{r}')}$$

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More than one particle:

Boson wave functions have to be symmetric:

$$b^+_k |0\rangle = |\phi_k\rangle$$

$$b^+_{k'} b^+_k |0\rangle = |\phi_{k'} \phi_k\rangle + |\phi_k \phi_{k'}\rangle$$

$$b^+_k b^+_{k'} |0\rangle = |\phi_k \phi_{k'}\rangle + |\phi_{k'} \phi_k\rangle \Rightarrow [b^+_k, b^+_{k'}] = 0$$

However, Fermions such as electrons have to be antisymmetric:

$$c^+_{k'} c^+_k |0\rangle = |\phi_{k'} \phi_k\rangle - |\phi_k \phi_{k'}\rangle$$

$$c^+_k c^+_{k'} |0\rangle = -c^+_{k'} c^+_k |0\rangle \Rightarrow c^+_k c^+_{k'} + c^+_{k'} c^+_k = 0$$

$$\Rightarrow \underbrace{\{c^+_k, c^+_{k'}\}}_{\text{anticommutator}} = 0 .$$ Also note $(c^+_k)^2 = 0$! This is just Pauli exclusion.

Bosons: $[b_k, b^+_{k'}] = \delta_{kk'}$ $\quad [b_k, b_{k'}] = 0$ $\quad \begin{cases} b^+_k |N_k\rangle = \sqrt{N_k+1}\, |N_k+1\rangle \\ b_k |N_k\rangle = \sqrt{N_k}\, |N_k-1\rangle \end{cases}$

Fermions: $\{c_k, c^+_{k'}\} = \delta_{kk'}$ $\quad \{c_k, c_{k'}\} = 0$ $\quad \begin{cases} c^+_k |N_k\rangle = \sqrt{1-N_k}\, |N_k+1\rangle \\ c_k |N_k\rangle = \sqrt{N_k}\, |N_k-1\rangle \end{cases}$

2nd quantization method: when dealing with many particles, it's more suitable to use a_k, a^+_k's instead of wave functions:

HW A4 problem 4: $\mathcal{H} = \sum_{i=1}^{N} T(\vec{r}_i) + \frac{1}{2} \sum_{i \neq j = 1}^{N} V(\vec{r}_i, \vec{r}_j)$

$$\Rightarrow \mathcal{H} = \sum_{k,k'} a^+_k \langle k|T|k'\rangle a_{k'} + \frac{1}{2} \sum_{k,k',k'',k'''} a^+_k a^+_{k'} \langle kk'|V|k''k'''\rangle a_{k'''} a_{k''}$$

note the ordering.

$[a_k, a^+_{k'}]_{\pm} = \delta_{k,k'}$ (Fermions: +)