Lecture #18: Equilibrium distribution of Fermions and Bosons, density of states.

Fermi's golden rule, from time-dependent perturbation theory.

Due to perturbation $V_{\text{int}}$ (e.g., electron-phonon, electron-phonon interaction)

$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | V_{\text{int}} | i \rangle|^2 \delta(E_f - E_i)$

$E_f = E_i + \hbar \omega_{\text{ph}}$

$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | \sum_{k,k'} A_{kk'} \gamma_{k'k} \gamma_k \gamma_k \gamma_{k'} | i \rangle|^2 \delta(E_f - E_i)$

(assuming that the actual rate will be an integration over this).

For example,

$V_{\text{int}} = \sum_{k,k'} A_{kk'} \gamma_{k'k} \gamma_k \gamma_k \gamma_{k'}$

Suppose the system is at $|i\rangle = |N_k = N_i \rangle$, a Fock state.

The transition rate into another Fock state $|f\rangle = |N_{k'} = N_f \rangle$ is given by:

$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | \sum_{k,k'} A_{kk'} \gamma_{k'k} \gamma_k \gamma_k \gamma_{k'} | i \rangle|^2 \delta(E_f - E_i)$

$= \sqrt{N_i} \sqrt{N_{f} + 1} A_{i f}$ for Bosons

$= \sqrt{N_i} \sqrt{1 - N_f} A_{i f}$ for Fermions

$\Gamma_{i \rightarrow f} = \frac{2\pi}{\hbar} \left( A_{i f} \right)^2 N_i \left\{ \begin{array}{ll} (1 + N_f) & \text{for Bosons} \\ (1 - N_f) & \text{for Fermions} \end{array} \right.$

*Stimulated emission*: The more photons there are in the final state, the easier it is the transition into it.

(opposite for Fermions)
What is the equilibrium occupation $f(E_K) = \langle N_K \rangle$ for Bosons and Fermions?

System equilibrates because of interparticle interactions,

$V_{\text{int}} = \frac{1}{2} \sum_{k_1, k_2, k_3, k_4} U_{k_1, k_2, k_3, k_4} a_{k_4}^+ a_{k_3}^+ a_{k_2} a_{k_1}$

No dangling with $k_1, k_2, k_3$ because of Conservation of momentum: $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$

Feynman diagram for the interaction

Focus on a particular state, say $k$.

$$\frac{dN_k}{dt} = \sum_{k_1, k_2, k_3} N_{k_2} N_{k_3} (1 \pm N_{k_1})(1 \pm N_k) - \sum_{k_1} N_k N_{k_1} (1 \pm N_{k_2})(1 \pm N_{k_3})$$

$$\frac{dN_k}{dt} = 0 \Rightarrow \left( E_{k_2} \right) f \left( E_{k_3} \right) \left[ 1 \pm f \left( E_{k_1} \right) \right] \left[ 1 \pm f \left( E_k \right) \right] = f \left( E_{k_1} \right) f \left( E_{k_3} \right) \left[ 1 \pm f \left( E_{k_2} \right) \right]$$

and $E_{k_1} + E_{k_2} = E_{k_2} + E_{k_3}$.

"Detailed balance": At equilibrium, amount going out equals amount going in!

Can be proven that

$\Rightarrow$ Only $f(E) = \frac{1}{e^{\frac{E}{k_B T}} - 1}$ satisfies this.

\[\Rightarrow\text{See next page}\]
Find $\alpha$ and $\beta$ from

\[ N = \sum_k f(E_k) = \int f(E) dE \]

\[ U = \sum_k E_k f(E_k) = \int E f(E) dE \]

\[ \ln \text{ in } \text{ limit } x > 1: \quad f(E) = e^{-\beta E} \]

\[ x = \frac{E}{RT} \quad \beta = \frac{1}{kT} \quad \Rightarrow \quad E = kT \ln \left( \frac{m}{\beta} \right) \]

The derivation gives an explanation why Bosons have the - and Fermions have the + sign in the definition here!

Define $f(E) = \frac{1}{g(E)+1}$ and plug into (*) to get:

\[ g(E_k)g(E_{k1}) = g(E_{k2})g(E_{k3}) \]

\[ \Rightarrow \quad h[g(E_k)] + h[g(E_{k1})] = h[g(E_{k2})] + h[g(E_{k3})] \]

\[ E_k + E_{k1} = E_{k2} + E_{k3} \]

Constraint of energy:

And these two equations are satisfied. We substitute only 

\[ h[g(E)] = (\alpha + \beta E) \quad \text{with } \alpha \text{ and } \beta \text{ constants} \]

\[ g(E) = e^{\alpha + \beta E} \]
Energy density for photons and phonons

\[ U = \int_{0}^{\infty} \text{d}E \cdot E \cdot f(E) \cdot D(E) \quad \text{and} \quad D(E) = \frac{1}{e^{\frac{\text{E}/kT}{\text{m}}/\text{e}} - 1} \]

because \( \mu = 0 \)

and photons and phonons are Bosons!

\[ C_V = \left( \frac{\partial U}{\partial T} \right)_V \]

Density of states per mode:

\[ \delta(E) \text{d}E = \frac{d^3k}{(2\pi)^3} = \frac{\eta \pi}{\text{V}} \frac{K^2}{V} \text{d}k \]

\[ = \frac{V}{z^\text{m}^2} \frac{k^2}{|\text{d}E|} \text{d}E \]

\[ D(E) = \frac{V}{z^\text{m}^2} \frac{E^2}{(\frac{\eta \pi}{\text{V}})^2} \frac{1}{(\frac{\eta \pi}{\text{V}})} \]

\[ \text{For photons, } N = \frac{5}{2}, \text{ and } 2 \text{ polarizations: } D_{\text{photons}}(E) = \frac{V m^3}{(\pi^2 \eta^2 c^3)} E^2 \]

\[ \text{For phonons, } 3 \text{ polarizations: } D_{\text{phonons}}(E) = \frac{3V}{z^\text{m}^2 (\frac{\eta \pi}{\text{V}})^3} E^2 \]
Black energy density in cavity:

\[ U(w) dw = \frac{h\nu}{e^{\frac{h\nu}{kT}} - 1} D_{\text{photon}}(\nu) = \frac{1}{e^{\frac{h\nu}{kT}} - 1} \left( \frac{V m^3}{T^2 c^3} \right) \nu^3 \]

\[ U_{\text{Total}} = \int_0^\infty U(w) dw = \left( \frac{V m^3}{kT} \right) \int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}} - 1} d\nu = \frac{V m^3}{kT} \left( \frac{4\pi^4}{15} \right) \int_0^\infty \frac{\nu^3}{e^{\frac{\nu}{kT}} - 1} d\nu = \frac{\pi^4}{15} \]

\[ \nu = \frac{3\nu}{kT} \]

\[ U_{\text{Total}} \propto T^4 \]

**Stefan-Boltzmann Law**

- **Black body radiation**: E.g., fit this to a star to determine the temperature of the star.

Heat capacity for phonons: Debye model

- Phonons are different from photons in the sense that they have a cut-off in \( k \): \( k = \frac{\pi}{a} \)
- Debye model is to assume a sharp cut-off, \( K = K_{\text{Debye}} = K_p \)

Therefore, \( D_{\text{photon}}(E) = \begin{cases} \frac{3V}{2\pi^2 \hbar^3} \frac{E^2}{N} & \text{if } E < \hbar \omega N K_p \\ 0 & \text{if } E > \hbar \omega N K_p \end{cases} \)}
To find \( k_D \), equate \( N_{\text{atom}} = \frac{4 \pi k_D^3}{(2 \pi)^2} = \frac{V}{6 \pi^2} k_D^3 \) \[ \Rightarrow k_D = \frac{6 \pi^2}{V} \left( \frac{N_{\text{atom}}}{V} \right) \]

\[ U_{ph} = 3 \int_{0}^{\infty} \frac{K_D}{k} \frac{k^3}{e^{\frac{k}{kT}} - 1} \]

At low \( T \), we can extend this integral to \( \infty \) because \( \frac{K_D}{kT} \gg 1 \) :

\[ U_{ph} = \int_{0}^{\infty} \frac{K_D}{k} \frac{k^3}{e^{\frac{k}{kT}} - 1} \]

\[ = \left( \frac{K_D}{kT} \right)^{4} \left( \frac{3V \frac{N}{2 \pi^2}}{kT} \right) \frac{4}{\pi} \propto T \]

\[ \Rightarrow C_V = \left( \frac{\partial U}{\partial T} \right)_V \propto T^3 \quad (T \ll T_{\text{Debye}} = \frac{\frac{4}{3}N\frac{K_D}{k_B}}{R}) \]

At high \( T \) \( T \gg T_{\text{Debye}} \) :

\[ U_{ph} = 3V \int_{0}^{k_B} \frac{K_D}{k} \frac{k^3}{\frac{2 \pi^2}{kT} + \frac{1}{T^2}} = \frac{3V k_B^2}{2 \pi^2} \int_{0}^{k_B} \frac{K_D}{k^2} k^2 \]

\[ = \frac{3V k_B^2}{2 \pi^2} k_D^3 = 3k_B T \left( \frac{V K_D^2}{6 \pi^2} \right) \]

\[ U_{ph} = 3N_{\text{atom}} k_B T \quad \left( 6N \text{ degrees of freedom} \right) \]

\[ \frac{C_V}{N} = \frac{3k_B}{2} \quad \left( \text{Debye - Dulong - Petit law} \right) \]

\[ \Rightarrow \text{show plots of pressure vs. temperature for real materials, require slides.} \]