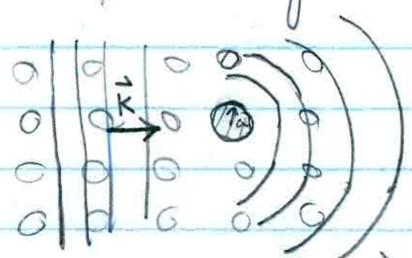
Lecture # 21 : Interactions with defects, Rayleigh scattering

Phonon-defect scattering



Classical energy of phonon field:

$$H = \int d^3r \, \frac{1}{2} \rho \, \omega^2 |\vec{u}|^2 \underset{\uparrow}{=} \int d^3r \, \frac{1}{2} \rho \, (\dot{\vec{u}})^2$$

$\dot{u} = \omega u$

A defect corresponds to a inhomogeneity in ρ, $\Delta\rho = (\rho' - \rho)$

$$H_{def-phon} = \frac{1}{2} \int d^3r \, (\Delta\rho) \, |\dot{\vec{u}}|^2$$

Use $\dot{\vec{u}}(\vec{r}) = -i \sum_{\vec{k},\lambda} \hat{\eta}_{k\lambda} \sqrt{\frac{\hbar \omega_k}{2\rho V}} \left(\hat{a}_{k\lambda} e^{i\vec{k}\cdot\vec{r}} - \hat{a}^{\dagger}_{\vec{k}\lambda} e^{-i\vec{k}\cdot\vec{r}} \right)$

$$\Rightarrow H_{def-phon} = \frac{1}{2} \sum_{\vec{k}_1\lambda_1} \sum_{\vec{k}_2\lambda_2} \int d^3r \left(\frac{\Delta\rho}{\rho}\right) \frac{\hbar\sqrt{\omega_1\omega_2}}{2V} \left(\hat{\eta}_{k_1\lambda_1} \cdot \hat{\eta}_{k_2\lambda_2}\right) \left(\hat{a}_{k_1\lambda_1} e^{i\vec{k}_1\cdot\vec{r}} - a^{\dagger}_{k_1\lambda_1} e^{-i\vec{k}_1\cdot\vec{r}}\right) \cdot$$

$$\cdot \left(\hat{a}_{k_2\lambda_2} e^{+i\vec{k}_2\cdot\vec{r}} - a^{\dagger}_{k_2\lambda_2} e^{-i\vec{k}_2\cdot\vec{r}}\right)$$

Inelastic $\begin{cases} a^{\dagger}a^{\dagger} & \text{Can't conserve energy unless defect "loses energy".} \\ a\,a & \text{will play a role if defect can be excited to a higher energy level} \end{cases}$

Elastic $\begin{cases} a^{\dagger}a & \\ a\,a^{\dagger} & \text{"Rayleigh scattering".} \end{cases}$

Assume $(\Delta\rho) = (\Delta\rho)_0\,\Theta(r-a)$ (defect is a sphere of radius a)

$$\mathcal{H}_{def-phon} = -\frac{1}{2}\sum_{\substack{k_1\lambda_1\\ k_2\lambda_2}} \frac{\hbar\sqrt{\omega_1\omega_2}}{2V}\frac{(\Delta\rho)_0}{\rho}\left(\hat{\eta}_{k_1\lambda_1}\cdot\hat{\eta}_{k_2\lambda_2}\right)\left[\hat{a}_{k_1}\hat{a}^{\dagger}_{k_2}\underbrace{\int d^3r\,\theta(r-a)\,e^{i\overbrace{|\vec{k}_1-\vec{k}_2|}^{|\Delta\vec{k}|}r\cos\theta}} + h.c.\right]$$

$$2\pi\int_{-1}^{1} d(\cos\theta)\int_0^a r^2dr\, e^{i|\Delta\vec{k}|r\cos\theta} = 4\pi\int_0^a dr\, r^2\,\frac{e^{i|\Delta k|r}-e^{-i|\Delta k|r}}{2i|\Delta\vec{k}|r} = \frac{4\pi}{|\Delta\vec{k}|}\int_0^a dr\, r\,\sin(|\Delta\vec{k}|r)$$

$$= \frac{4\pi}{|\Delta\vec{k}|}\left(\frac{-d}{d|\Delta k|}\right)\underbrace{\int_0^a dr\,\cos(|\Delta k|r)}_{\frac{\sin[|\Delta k|a]}{|\Delta\vec{k}|}} = \frac{4\pi}{|\Delta\vec{k}|}\,\frac{-a\cos(|\Delta\vec{k}|a)|\Delta\vec{k}| + \sin(|\Delta k|a)}{|\Delta\vec{k}|^2}$$

when $|\Delta\vec{k}|a \ll 1$ $(\lambda \gg a) \Rightarrow$ $= \dfrac{4\pi}{|\Delta k|^3}\left[-|\Delta k|a\left(\cancel{1}-\dfrac{\Delta k^2a^2}{2}\right) + \cancel{(\Delta k a)} - \dfrac{(\Delta k a)^3}{3!} + \Delta k^4\right]$

(Born approx.)

$$= \frac{4\pi}{3}a^3 + \mathcal{O}(\Delta k)^4$$

$$\mathcal{H}_{def-phon} = -\frac{1}{2}\,\frac{(\Delta\rho)_0}{\rho}\,\frac{2\pi}{3}a^3\sum_{\substack{\vec{k}_1,\lambda_1\\ \vec{k}_2,\lambda_2}}\frac{\hbar\sqrt{\omega_1\omega_2}}{V}\left(\hat{\eta}_{k_1\lambda_1}\cdot\hat{\eta}_{k_2\lambda_2}\right)\underbrace{\left(\hat{a}_{k_1\lambda_1}\hat{a}^{\dagger}_{k_2\lambda_2} + \hat{a}^{\dagger}_{k_2\lambda_2}\hat{a}_{k_1\lambda_1}\right)}_{\left(2\hat{a}^{\dagger}_{k_2\lambda_2}\hat{a}_{k_1\lambda_1} + \underbrace{\delta_{k_1\lambda_1,k_2\lambda_2}}_{\text{const.) drop}}\right)}$$

$$\boxed{\mathcal{H}_{def-phon} = -\left(\frac{\Delta\rho_0}{\rho}\right)\left(\frac{2\pi}{3}a^3\right)\sum_{\substack{\vec{k}_1,\lambda_1\\ \vec{k}_2,\lambda_2}}\frac{\hbar\sqrt{\omega_1\omega_2}}{V}\left(\hat{\eta}_{k_1\lambda_1}\cdot\hat{\eta}_{k_2\lambda_2}\right)\hat{a}^{\dagger}_{k_2\lambda_2}\hat{a}_{k_1\lambda_1}}$$

Note: $\hbar\omega_{k_1} = \hbar\omega_{k_2}$ (elastic) but $\vec{k}_1 \neq \vec{k}_2$ (momentum is not conserved due to defect's recoil!)

Continue on Nov 22

↓

use Fermi's golden rule to find the mean scattering time due to Rayleigh:

$$\frac{1}{\tau_{\vec{k}\lambda}} = \frac{1}{N_{k\lambda}} \frac{2\pi}{\hbar} \left(\frac{\Delta\rho}{\rho}\right)^2 \left(\frac{2\pi}{3}a^3\right)^2 \sum_{\vec{k}',\lambda'} \frac{\hbar^2}{V^2}(\omega_{k\lambda}\omega_{k'\lambda'}) \left(\hat{n}_{k\lambda}\cdot\hat{n}_{k'\lambda'}\right)^2 N_{k\lambda}\left(N_{k'\lambda'}+1\right) \times$$

$$\times\, \delta\left(\hbar\omega_k - \hbar\omega_{k'}\right)$$

↑ scattering time for phonon $\vec{k}\lambda$

Assume $\lambda = \lambda' = L$ so that $\left(\hat{n}_k \cdot \hat{n}_{k'}\right) = \cos(\theta')$

$$\frac{1}{\tau_k} = \frac{2\pi}{\hbar}\left(\frac{\Delta\rho}{\rho}\right)^2 \left(\frac{2\pi}{3}a^3\right)^2 \frac{\hbar^2 \omega_k^2}{V^2}\left(N_k+1\right)(2\pi)\underbrace{\int_{-1}^{1} d(\cos\theta)\,(\cos(\theta))^2}_{\left.\frac{(\cos\theta)^3}{3}\right|_{\cos\theta=-1}^{\cos\theta=1}} \int_0^\infty \frac{dk'(k')^2}{\frac{(2\pi)^3}{V}} \frac{\delta(k-k')}{\hbar v}$$

$$\frac{1}{\tau_k} = \left(\frac{\Delta\rho}{\rho}\right)^2 \frac{a^6}{9} \frac{v^2k^2}{V}\left(N_k+1\right)\frac{4\pi}{3}\frac{k^2}{v}$$

$$\frac{1}{\tau_k} = \left(\frac{\Delta\rho}{\rho}\right)^2 \frac{4\pi}{27} a^6 \frac{1}{Vv^3}\left(N_k+1\right)\omega^4$$

If there are N_d defects in volume V: $n_d = \frac{N_d}{V}$ and

$$\boxed{\frac{1}{\tau_k} = \left(\frac{\Delta\rho}{\rho}\right)^2 \frac{4\pi}{27} a^6 \left(N_k+1\right)\frac{n_d}{v^3}\,\omega^4}$$

Also define a cross section σ from $\frac{1}{\tau} = \sigma n_d v$

$$\Rightarrow \boxed{\sigma = \frac{4\pi}{9}\left(\frac{\Delta\rho}{\rho}\right)^2 \left(N_k+1\right)\frac{\omega^4}{v^4}a^6}$$

$\Rightarrow$ Mean free path $l = v\tau = \frac{1}{\sigma n_d} \propto \omega^{-4}$

Submarines and whales use this to communicate

- Acoustic "sound" with $f = 0 - 10\,kHz \Rightarrow l \sim 1\,Km$!
- Ultra sound ($f = 1 - 10\,MHz$), $l \sim mm - cm$ (Ultrasound absorption is great to image organs in the body!) You can see babies!
- High frequency phonons (zone edge acoustic, optical) have $l \sim nm \Rightarrow$ They move diffusively instead of ballistically

Photon - defect scattering

$$H_{def-phot} = -\frac{1}{2}\int d^3r\,(\vec{P}\cdot\vec{E})$$

For a dielectric sphere of radius a, and index of refraction n: $\quad \vec{P} = 3\varepsilon_0\left(\frac{n^2-1}{n^2+2}\right)\Theta(a-r)\,\vec{E}$

Use $\vec{E}$ in terms of photon creation and annihilation operators to find:

$$\boxed{H_{def-ph} = -\left(\frac{n^2-1}{n^2+2}\right) 2\pi a^3 \sum_{k_1,\lambda_1}\sum_{k_2,\lambda_2}(\hat{n}_{k_1\lambda_1}\cdot\hat{n}_{k_2\lambda_2})\,\frac{\hbar\sqrt{\omega_1\omega_2}}{V}\,\hat{a}^{+}_{k_1\lambda_1}\hat{a}_{k_2\lambda_2}}$$

Similarly,

$$\boxed{\frac{1}{\tau_k} = \frac{8\pi}{3}\left(\frac{n^2-1}{n^2+2}\right)^2\frac{a^6}{c^3}\,n_d\,(1+N_k)\,\omega^4}$$

$$\boxed{\sigma = \frac{8\pi}{3}\left(\frac{n^2-1}{n^2+2}\right)^2\frac{a^6}{c^4}\,(1+N_k)\,\omega^4}$$

This explains why the sky is blue: High frequency light (blue) scatters much stronger than low frequency (red).
$\Rightarrow$ The color of the atmosphere is a result of scattering from "particulates" with radius 50 nm and $n=2$! (local fluctuations of the gases)

Compare to Classical EM result: It's the same when $N_k = 0$!

$\Rightarrow$ Stimulated emission is a true quantum effect.