Phys 507B - Solid State Physics II

Assignment 4: Coherence, correlation, and noise. *Due March 29th*

1. *Free induction decay (FID) in the pure dephasing limit.*

   Consider a spin evolving according to the Hamiltonian

   $$\mathcal{H} = \omega_0 S_z + \Delta(t) S_z, \tag{1}$$

   where $\omega_0 = g\mu B_0/\hbar$ is the precession frequency due to an external field along the z-direction, and $\Delta(t)$ models the effect of the environment on the spin. We are assuming that the environment corresponds to a simple fluctuation of the field along the z-direction; obviously, this does not cause energy relaxation (why?), so that $T_1 = \infty$. However, it does cause decoherence, $T_2 < \infty$.

   (a) In a free induction decay experiment, a $\pi/2$ pulse is applied to the spin so that the spin precesses in plane. Let's call the state of the spin right after this $\pi/2$ pulse $\rho_0 = \rho(t=0)$. Show that in this case,

   $$\langle S_+(t)\rangle = \mathrm{Tr}\left\{S_+\rho(t)\right\} = \langle S_+(0)\rangle\, \mathrm{e}^{-i\omega_0 t}\mathrm{e}^{i\theta(t)}, \tag{2}$$

   where

   $$\theta(t) = \int_0^t dt' \Delta(t'). \tag{3}$$

   Here $S_+ = S_x + iS_y$ is the spin ladder operator. *Hint: Find the evolution operator $\mathcal{U}(t)$, and then use the following identities to simplify $\langle S_+\rangle$:*

   $$\begin{aligned} \mathrm{e}^{i\frac{\theta}{2}\sigma_z}\sigma_x \mathrm{e}^{-i\frac{\theta}{2}\sigma_z} &= \cos(\theta)\sigma_x - \sin(\theta)\sigma_y, \\ \mathrm{e}^{i\frac{\theta}{2}\sigma_z}\sigma_y \mathrm{e}^{-i\frac{\theta}{2}\sigma_z} &= \sin(\theta)\sigma_x + \cos(\theta)\sigma_y. \end{aligned}$$

   (b) The expression derived in (a) is simple enough for us to go beyond the Bloch-Wangsness-Redfield approximation; in the following, we will solve the free induction decay problem exactly for general noise $\tilde{S}_\Delta(\omega)$. Assume Δ is "Gaussian noise", i.e., the distribution of $\Delta(t')$ is a Gaussian at any given time t'. Argue that the distribution of $\theta(t)$ will also be Gaussian, given by

   $$P(\theta(t)) = \frac{1}{\sqrt{2\pi}\sigma_t}\mathrm{e}^{-\frac{\theta^2}{2\sigma_t^2}}. \tag{4}$$

   Find σ_t^2 in terms of the correlation function $S_\Delta(\tau)$, and in terms of the noise spectrum $\tilde{S}_\Delta(\omega)$.
   Hint: $\sigma_t^2 = \langle[\theta(t)]^2\rangle$; assume "stationarity", and write σ_t^2 as a double time integral of $S_\Delta(\tau)$. Next, insert the noise spectrum and convert to a single integral over frequency.

(c) Now find the ensemble average of Eq.(2) over possible realizations of the time series $\Delta(t')$ (i.e., find $\langle\langle S_+(t)\rangle\rangle$). Write your final result as

$$\langle\langle S_+(t)\rangle\rangle = \langle S_+(0)\rangle\, \mathrm{e}^{-t^2\int_{-\infty}^{\infty} d\omega \tilde{S}(\omega)\mathcal{F}(\omega t)}, \tag{5}$$

where $\mathcal{F}(\omega t)$ is a filter function. Which frequency region of the noise spectrum causes coherence decay? Sketch the filter function.

2. *Spin echo.*

The spin echo is one of the most important tricks of magnetic resonance spectroscopy. In this problem you will show how the spin echo enables one to remove inhomogeneous broadening in order to measure the intrinsic decoherence time T_2.

(a) Using the same model Hamiltonian Eq. (1), consider the following experiment: We apply a $\pi/2$ pulse and get a state ρ_0; we let this state evolve for time τ, and then we apply a π-pulse about the x-axis,

$$\hat{X}_\pi = -i\sigma_x. \tag{6}$$

Then the system is allowed to evolve again. Find the time evolution operator $\mathcal{U}(t)$ that describes the state at times $t > \tau$ (all times after the π-pulse).

(b) Compute the resulting $\langle S_+(t)\rangle$ as a functional of the time series $\Delta(t')$ (no need to take ensemble average over Gaussian noise here – it can be done, but it's a bit complicated).

(c) Assume the noise is "static", i.e., $\Delta(t') = \Delta_0$. What happens when $t = 2\tau$? Give a geometrical explanation, sketching what happens to the spin in the rotating frame. Explain how the spin echo experiment is able to remove inhomogeneous broadening.

3. *Free induction decay (FID) for Gaussian noise with Lorentzian spectrum.*

(a) Using the result derived in Problem 1(c) above, find the time dependence of FID for the commonly encountered spectrum

$$\tilde{S}_\Delta(\omega) = \frac{\Delta_0^2\tau_c}{\pi}\frac{1}{(\omega\tau_c)^2+1}. \tag{7}$$

Hint: You can find the FID decay integrating the noise times the filter function, but you will have to use complex integration (Cauchy's theorem); alternatively, you may find it easier to compute the double time integral over the corresponding correlation function $S_\Delta(\tau) = \Delta_0^2 e^{-|\tau|/\tau_c}$.

(b) How does the decay look like when $t \ll \tau_c$? Interpret this as inhomogeneous broadening, because the noise "looks static" and could in principle be reverted using a spin echo, as in Problem 2. Find the time scale T_2^* that gives the $1/e$ decay of the FID.

(c) How does the decay look like when $t \gg \tau_c$? Interpret this as homogeneous broadening (because Δ changes too fast to be reverted by an echo) and find the time scale T_2 that gives $1/e$ decay of the FID. When $\tau_c \to 0$, does T_2 increases or decreases? Interpret this result. This phenomena is called "motional narrowing" by people who do liquid state NMR.

4. *Spectral decomposition of noise.*

Consider the formal definition of thermal equilibrium noise for a quantum observable $\hat{A}$,

$$\tilde{S}_A(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau \mathrm{e}^{i\omega\tau} \left\langle \hat{A}^\dagger(\tau)\hat{A}(0)\right\rangle_{\text{Thermal}}. \tag{8}$$

(a) Give the proper definition of the thermal correlation function inside the integral.

(b) Assume that the energy eigenstates of the Hamiltonian are known, $\mathcal{H}|\alpha\rangle = E_\alpha|\alpha\rangle$. Write down a formal expression for Eq. (8) that involves these energy eigenstates and eigenenergies explicitly.

(c) Prove the general "detailed balance" result $\tilde{S}_A(-\omega) = \mathrm{e}^{-\frac{\hbar\omega}{k_B T}}\tilde{S}_A(\omega)$.