Phys 507B - Solid State Physics II

Assignment 5: Magnetism. *Due April 21st*

1. *Phenomenological (Landau) theory of Ferromagnetism.*

   The free energy F of a magnetic system may be written as the volume integral of a free energy density:
$$F = \int d^3x \mathcal{F}\left[M_\mu(\boldsymbol{x}), \partial_\nu M_\mu(\boldsymbol{x})\right]. \tag{1}$$
   Here the order parameter for the phase transition is the three component "Classical" vector $\boldsymbol{M}(\boldsymbol{x}) = M_\mu(\boldsymbol{x})\hat{\boldsymbol{e}}_\mu$, where $M_\mu(\boldsymbol{x})$ is the μ-th component of the magnetization at position $\boldsymbol{x} = x_\nu\hat{\boldsymbol{e}}_\nu$ in the material, and $\partial_\nu M_\mu = \frac{\partial M_\mu}{\partial x_\nu}$ is one of the spatial derivatives of M_μ.

   Consider the following model free energy for an isotropic ferromagnet,
$$\mathcal{F} = \frac{\alpha}{2}\sum_{\mu,\nu}\left(\partial_\nu M_\mu\right)^2 + \frac{\beta}{2}M^2 + \frac{\gamma}{4}M^4 - \boldsymbol{H}\cdot\boldsymbol{M}, \tag{2}$$
   where α and γ are always positive.

   (a) The ground state of the system is found by minimizing F. Show that $\boldsymbol{M}_0$ will be an extremum when it satisfies
$$\frac{\delta\mathcal{F}}{\delta\boldsymbol{M}} = \frac{\partial\mathcal{F}}{\partial M_\mu}\hat{\boldsymbol{e}}_\mu - \partial_\nu\frac{\partial\mathcal{F}}{\partial(\partial_\nu M_\mu)}\hat{\boldsymbol{e}}_\mu = \frac{\partial\mathcal{F}}{\partial\boldsymbol{M}} - \nabla\cdot\frac{\partial\mathcal{F}}{\partial(\nabla\boldsymbol{M})} = 0. \tag{3}$$
   The generalized derivative $\frac{\delta\mathcal{F}}{\delta\boldsymbol{M}}$ is known as the functional derivative, a derivative in the space of functions. Of course, you should always remember that some extrema are maxima (unstable) and other extrema are minima (stable). Only the latter will be a physical phase of matter.

   Hint: Find this relationship from the $\delta F = 0$ condition. The calculation is just like the minimization of the action in classical mechanics, you will need to use integration by parts to get the divergence term.

   (b) Assume $\boldsymbol{H} = 0$ for now. Find the ground state $\boldsymbol{M}_0$ that minimizes F, for (i) $\beta < 0$, and (ii) for $\beta > 0$. Show that this state is ferromagnetic only when $\beta < 0$. For $\beta > 0$, argue that the state is paramagnetic. We set $\beta = \beta_0(T - T_c)$, where T_c is the transition temperature.

   (c) For $T > T_c$, show that the static magnetic susceptibility is given by $\chi = \frac{1}{\beta_0(T-T_c)}$.

   (d) For $T < T_c$, show that the static magnetic susceptibility is given by $\chi = \frac{1}{2\beta_0(T_c-T)}$.

2. *Classical derivation of the Landau-Lifshitz equation of motion*

So far we have only considered the dependences of the magnetization on space and temperature. When we drive the magnet with a time dependent magnetic field, the magnetization will precess. Here you will derive, under quite general assumptions, the equation that describes this time dependence.

Consider a classical system of N particles, described by coordinates $\boldsymbol{r}_i$ and canonical momenta $\boldsymbol{p}_i$, $i = 1, \ldots, N$. Assume that the dynamics of the system is governed by the Hamiltonian $\mathcal{H}(\boldsymbol{r}_i, \boldsymbol{p}_i)$. Consider two macroscopic observables $A(\boldsymbol{r}_i, \boldsymbol{p}_i)$ and $B(\boldsymbol{r}_i, \boldsymbol{p}_i)$. A useful concept is the *Poisson bracket*:

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial \boldsymbol{r}_i} \cdot \frac{\partial B}{\partial \boldsymbol{p}_i} - \frac{\partial A}{\partial \boldsymbol{p}_i} \cdot \frac{\partial B}{\partial \boldsymbol{r}_i} \right). \tag{4}$$

It turns out that the Poisson bracket is the classical version of the commutator in quantum mechanics. You may convince yourself that all familiar properties of the commutator in quantum mechanics also apply to the Poisson bracket. The classical to quantum transition is obtained by $\{,\} \to \frac{1}{i\hbar}[,]$ (for example, note that $\{x, p_x\} = 1$.

(a) Prove that

$$\frac{dA}{dt} = \{A, H\} \tag{5}$$

Hint: Use the derivative chain rule and substitute Hamilton's equations of motion.

(b) Define the angular momentum density at position $\boldsymbol{x}$ as

$$\boldsymbol{L}(\boldsymbol{x}) = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i \delta(\boldsymbol{x} - \boldsymbol{r}_i). \tag{6}$$

Show that

$$\{L_\alpha(\boldsymbol{x}), L_\beta(\boldsymbol{y})\} = \epsilon_{\alpha\beta\gamma} L_\gamma(\boldsymbol{x}) \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{7}$$

Here $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol.

(c) The magnetization density is given by $\boldsymbol{M}(\boldsymbol{x}) = \gamma_e \boldsymbol{L}(\boldsymbol{x})$, where $\gamma_e = \frac{ge}{2m_e c}$ is the gyromagnetic ratio (of course, in addition to classical angular momentum, we must add the spin contribution to $\boldsymbol{M}$ – It turns out that all arguments and results described below are absolutely general and apply equaly well to quantum spin).

Show that

$$\begin{aligned} \{M_\alpha(\boldsymbol{x}), M_\beta(\boldsymbol{y})\} &= \gamma_e \epsilon_{\alpha\beta\gamma} M_\gamma(\boldsymbol{x}) \delta(\boldsymbol{x} - \boldsymbol{y}), & (8) \\ \{M_\alpha(\boldsymbol{x}), \partial_\nu M_\beta(\boldsymbol{y})\} &= \gamma_e \epsilon_{\alpha\beta\gamma} M_\gamma(\boldsymbol{x}) \frac{\partial}{\partial y_\nu} \delta(\boldsymbol{x} - \boldsymbol{y}). & (9) \end{aligned}$$

(d) Using the general result of item (a), and the Poisson brackets of item (c), derive the Landau-Lifshitz equation of motion:

$$\frac{\partial \boldsymbol{M}(\boldsymbol{x})}{\partial t} = -\gamma_e \boldsymbol{M}(\boldsymbol{x}) \times \frac{\delta \mathcal{F}}{\delta \boldsymbol{M}(\boldsymbol{x})}. \tag{10}$$

Hint: In place of H, use the free energy $F = \int d^3y \mathcal{F}$. Assume that the density $\mathcal{F}$ is analytical on M_μ and $\partial_\nu M_\mu$, i.e., it can be expanded as a power series on these observables.

3. *Phenomenological theory of spin waves in a simple ferromagnet.*

The equation of motion derived in problem 4 can be used to find the excited states of the simple ferromagnet described in problem 3. Use the free energy Eq. (2), with an applied magnetic field $\boldsymbol{H} = H_0\hat{z}$.

(a) Assume a spin wave state $\boldsymbol{M} = M_0\hat{z} + \delta\boldsymbol{M}(\boldsymbol{x}, t)$. This is a perturbation over the ground state $\boldsymbol{M}_0$ derived in problem 3. Assume that the perturbation $\delta\boldsymbol{M}$ is in the xy plane (it has to be – look at the Landau-Lifshitz equation of motion). Show that the linearized equation of motion [after dropping terms of order $(\delta M)^2$] becomes

$$\frac{\partial \delta\boldsymbol{M}}{\partial t} = \gamma_e \alpha \boldsymbol{M}_0 \times \nabla^2 \delta\boldsymbol{M} - \gamma_e \boldsymbol{H} \times \delta\boldsymbol{M}. \tag{11}$$

Search for plane wave solutions $\delta\boldsymbol{M}(\boldsymbol{x}, t) = \delta\boldsymbol{M}\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ and find the magnon dispersion, $\omega(k)$.

(b) *Spin resonance.* Now assume $\boldsymbol{H} = H_0\hat{z} + \delta\boldsymbol{H}\mathrm{e}^{-i\omega t}$, where $\delta\boldsymbol{H}$ is in the xy plane. Find the dynamic susceptibility matrix defined by $\delta\boldsymbol{M} = \boldsymbol{\chi}(\omega)\cdot\delta\boldsymbol{H}$. Show that it diverges when ω equals the magnon frequency at $k = 0$.

4. *Symmetries of the "quantum" Heisenberg Hamiltonian.*

Consider the Heisenberg Hamiltonian

$$\mathcal{H} = -J\sum_{j,\delta} \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+\delta} - 2\mu_0 H_0 \sum_j S_{jz}. \tag{12}$$

The total spin operator is given by $\boldsymbol{S} = \sum_j \boldsymbol{S}_j$.

(a) Prove that $[S_z, \mathcal{H}] = 0$.

(b) Prove that $[S^2, \mathcal{H}] = 0$.